



## REPRESENTATION PATTERNS OF EUCLIDEAN GEOMETRY TASKS IN SECONDARY SCHOOL MATHEMATICS TEXTBOOKS

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**Abstract:** Task representations are crucial in mathematics teaching and learning, allowing students to develop a deep understanding and meaningful connections between mathematical concepts. Representation encompasses how a problem or situation is presented, including the setting, data, and task. Four distinct representation forms exist: pure mathematical tasks, verbal tasks, visual tasks, and combined tasks. Effective representation facilitates students' understanding, interpretation, and communication of mathematical concepts, making it a crucial aspect of teaching and learning mathematics. This qualitative case study examines Euclidean geometry tasks in three FET mathematics textbook series, investigating how textbooks provide opportunities to learn Euclidean geometry through representation. The textbook content analysis of tasks was conducted based on the classification of problem types established in Zhu and Fan's theoretical framework, which focuses on task representation. The findings revealed that combined representations overwhelmingly dominated the Euclidean geometry tasks, while verbal representations occurred only to a limited extent. No tasks were presented exclusively in purely mathematical or visual forms, reflecting the inherently spatial nature of Euclidean geometry. Although the predominance of combined representations appropriately supports the communication of geometric relationships, the limited diversity of representation forms may constrain students' opportunities to engage with different modes of mathematical reasoning. The findings have implications for textbook development, classroom practice, and future research on Opportunity to Learn in mathematics education..

**Keywords:** Euclidean Geometry; Mathematics Education; Opportunity to Learn; Task representation; Textbook Analysis

### 1. Introduction

Task representation is a crucial aspect of constructing and abstracting mathematical knowledge, playing a vital role in fostering students' understanding and advancing their cognitive abilities, particularly in geometry education (Clements et al., 2018). Effective representation facilitates the development of students' spatial reasoning, visual literacy, and critical thinking skills, ultimately enriching their comprehension of geometric concepts. Mathematical representation is a fundamental tool for articulating and conveying mathematical concepts clearly and concisely (Zhu & Fan, 2006). By translating abstract mathematical ideas into a comprehensible format, mathematical representation enables effective communication of mathematical knowledge (National Council of Teachers of Mathematics, 2000; Tripathi, 2008). The primary function of mathematical representation is to provide visual, verbal, symbolic, or combined frameworks for understanding mathematical concepts, thereby fostering deeper and more nuanced conceptual understanding (Polat, 2020).

The type of representation incorporated into mathematics tasks has a profound influence on students' cognitive processes, shaping how they interpret, reason about, and solve mathematical problems (Fan & Zhu, 2007; Hwang et al., 2020; Yang et al., 2017). Different representation forms either facilitate or constrain students' understanding, reasoning, and problem-solving abilities (Bütüner, 2021; Wang & Yang, 2016). Consequently, the thoughtful selection and design of task representations are fundamental to promoting effective learning and developing students' geometric thinking (Alkhateeb,

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2018). From an Opportunity to Learn (OTL) perspective, the representations embedded in textbook tasks constitute an important dimension of the learning opportunities available to students. OTL extends beyond curriculum coverage to encompass the quality and diversity of learning experiences afforded through instructional materials. In mathematics textbooks, these opportunities are reflected in how concepts are represented and in the range of cognitive processes that different representation forms require students to engage in.

Euclidean geometry is a fundamental component of the mathematics curriculum because of its extensive real-world applications (Dimla, 2018). It explores the properties and relationships of points, lines, surfaces, and solids within a two-dimensional plane (Jones & Tzekaki, 2016). The study of Euclidean geometry cultivates important mathematical competencies, including visualisation, critical thinking, intuition, problem-solving, conjecturing, deductive reasoning, logical argumentation, and proof (Jones, 2000). These competencies not only deepen students' mathematical understanding and spatial reasoning but also prepare them for careers in Science, Technology, Engineering and Mathematics (STEM) fields such as architecture, engineering, and computer graphics (Clements et al., 2018).

Despite its importance, Euclidean geometry continues to present considerable challenges for both teachers and students in secondary schools (Abakah & Brijlall, 2023; Bansilal, 2019; Hlalele, 2020; Naidoo & Hajaree, 2021). Research has consistently shown that students experience difficulties with fundamental geometric concepts such as angles, triangles, and quadrilaterals (Chiphambo & Feza, 2020; Machisi & Feza, 2021), as well as higher-order concepts involving deductive reasoning and proof (Barut & Retnawati, 2020; Barut & Wijaya, 2021; Hatziminadakis, 2018; Kalyankar, 2019). Alex and Mammen (2018) found that many South African learners who had completed geometry instruction were unable to demonstrate proficient proof construction and geometric reasoning, with most operating at the lower levels of the van Hiele model of geometric thinking. Similarly, Atebe (2008) reported that only 2% of Nigerian students and 3% of South African students had reached van Hiele Level 2 (informal deduction), indicating that most learners were not yet ready for formal deductive geometry. These findings are further corroborated by the Department of Basic Education Diagnostic Report (DBE, 2024), which highlights persistent learner difficulties with geometry tasks requiring higher-order reasoning.

Mathematics textbooks play a pivotal role in shaping students' opportunities to learn because they serve as the primary curriculum resource in many classrooms (Mthethwa et al., 2024; Ogbonnaya, 2021; O'Keeffe, 2013; Valverde et al., 2002). Through the tasks they present, textbooks determine not only the mathematical content students encounter but also how that content is represented and the kinds of thinking students are expected to perform. Consequently, textbook tasks constitute an important indicator of students' opportunity to learn. When textbooks provide diverse representation forms, they create opportunities for students to interpret, connect, and translate mathematical ideas across multiple representations, thereby strengthening conceptual understanding, mathematical communication, and problem-solving. Conversely, a narrow range of representations may limit students' exposure to different modes of mathematical thinking and constrain the development of essential geometric competencies.

Although some studies have investigated representation forms in mathematics textbooks across different countries (Hidayah & Forgasz, 2020; Yang et al., 2017; Yang & Sianturi, 2022), limited research has specifically examined the learning opportunities that South African secondary school mathematics textbooks provide through the representation of Euclidean geometry tasks. Understanding these opportunities is particularly important given the persistent challenges learners experience in geometry and the central role textbooks play in classroom instruction. Examining the forms of representation, therefore, provides insight not only into how geometry tasks are presented but also into the extent to which textbooks create opportunities for students to develop the diverse cognitive skills required for meaningful learning of Euclidean geometry.

This study investigated how different task representation forms are used to provide opportunities to learn Euclidean geometry across three Further Education and Training (FET) mathematics textbook series. Specifically, the study examines the extent to which Euclidean geometry tasks are presented in

pure mathematical, verbal, visual, and combined representation forms. The research question guiding the study is: How are different task representation types used to provide opportunities to learn Euclidean geometry?

## 2. Literature Review

Textbooks are a vital component of curriculum materials, playing a pivotal role in teaching and learning mathematics (Mthethwa et al., 2024; Ogbonnaya, 2021; O'keeffe, 2013; Valverde et al., 2002). Notably, textbooks utilise diverse forms of mathematical representation to convey complex mathematical concepts to their audience (Alkhateeb, 2018). Furthermore, research emphasises the significance of representation forms within mathematical tasks. Lee and Guajardo (2021) contend that these representation forms are instrumental in the teaching and learning of mathematical concepts, transcending mere procedural knowledge. Similarly, Utomo (2021) views representation forms as crucial for fostering a deeper comprehension of mathematical knowledge and skills. The effectiveness of textbooks in promoting deep understanding and meaningful learning of geometry depends on the quality of representation used to convey geometric concepts (Alkhateeb, 2018; Fan et al., 2018; Mainali, 2021). Consequently, the effective design and utilisation of representation forms in textbooks are essential considerations for mathematics teaching and learning (Pepin et al., 2013).

Effective representations improve comprehension, problem-solving skills, and mathematical literacy (Liu et al., 2021). According to Niss (2003), employing multiple mathematical representation features is essential for facilitating the organisation, documentation, and communication of mathematical ideas. This approach enables the articulation of complex mathematical concepts through diverse representation forms, enhancing clarity, precision, and comprehension.

Numerous researchers have employed various frameworks to classify mathematical tasks, reflecting task design's complexity and multifaceted nature. One framework proposed by Zhu and Fan (2006) categorises mathematical tasks into four distinct categories based on their representational forms: pure-mathematical tasks, verbal tasks, visual tasks, and combined tasks. Tasks whose stem exclusively comprise mathematical expressions without any accompanying text or context are categorically defined as tasks in pure mathematical form (Zhu & Fan, 2006). Tasks whose stem consists exclusively of written words, devoid of mathematical expressions or symbols, are classified as tasks in verbal form (Yang & Sianturi, 2022). Yang (2020) classifies tasks whose stems consist exclusively of visual elements, such as figures, pictures, graphs, charts, tables, diagrams, maps, and other non-textual representations, as visual form tasks. Tasks with stems presented in a multimodal format, combining two or more representational forms, are categorised as combined-form tasks (Zhu & Fan, 2006). Zhu and Fan's (2006) framework provides a structured approach to understanding the diverse ways mathematical tasks can be represented and has implications for task design, instructional practices, and student learning outcomes.

Using multiple representations in mathematics means employing appropriate and effective representations in observable ways to depict mathematical concepts (Powell et al., 2022). Several studies have recommended effectively integrating multiple modes of representation forms in the mathematics classroom to improve students' conceptual understanding (Alkhateeb, 2018; Finamore et al., 2021; Liu et al., 2021; Yang et al., 2017). According to Pedersen et al. (2021), using tasks involving multiple representations enables students to comprehend mathematical concepts. Alkhateeb (2018) emphasises the importance of integrating symbolic and verbal representations in textbooks and teachers' effective use of these representations in promoting students' mathematical understanding. However, by representing abstract concepts through verbal and symbolic combinations of real objects, students can truly understand how mathematics is applied in real-life situations. This approach enhances their engagement and fosters deeper learning by connecting theoretical knowledge with practical experiences. By bridging the gap between abstract ideas and tangible applications, teachers can cultivate a more meaningful understanding of mathematics among their students.

Research has consistently demonstrated the efficacy of diverse representation forms in mathematics education. For instance, Bansilal (2019) found that employing various representation forms significantly enhances students' comprehension and development of geometry concepts. Similarly,

Booyesen and Westaway (2022) argue that the strategic use of different representation forms facilitates effective communication and clarification of mathematical concepts, ultimately benefiting mathematics instruction.

In this context, mathematics textbooks play a pivotal role as a medium for students to learn mathematical ideas presented in various representation forms. By leveraging these representation forms, textbooks have the potential to promote deeper understanding, improve learning outcomes, and foster a more engaging and inclusive mathematics education experience. As such, textbooks should provide a variety of task representations, enabling students to actively engage in the learning process. Clements et al. (2018) suggest that using multiple modes of representation for geometry concepts can assist teachers in their instructional transitions, thereby enhancing students' comprehension and success.

Anthony and Walshaw's (2009) seminal review of research on representations and transitions revealed that mathematical representations serve as a crucial bridge connecting disparate mathematical ideas and concepts. Furthermore, the findings underscored the significance of representations in facilitating students' comprehension of the intricate interrelationships between mathematical concepts. By providing opportunities for students to explore diverse representations of each concept, representations enable learners to develop a deeper understanding of the underlying mathematical structures and relationships. Similarly, Pedersen et al. (2021) reviewed numerous studies on representations and transitions and concluded that representations play a pivotal role in mathematical activities. The study's findings revealed a relationship between the mathematical concepts and the presentation form used in the question. This correlation encourages students to understand the various connections among these concepts by exploring different representations of these concepts.

Yang and Sianturi (2022) investigated task design in elementary mathematics textbooks across five countries, focusing on the students' understanding of the textbook series. This study's research design involved a content analysis of mathematics problems organised around four primary dimensions. These dimensions comprised representational forms, cognitive demand levels, contextual features, and types of responses. An analysis of the representation forms category indicates that Finnish textbooks contained the largest percentage of symbolic representations (75.03%) compared to the other four countries. Conversely, Malaysia had the most questions presented in verbal form (41.35%) and combination forms (29.11%). Another important finding was that, when comparing the four forms, Singapore textbooks presented more tasks in symbolic forms (52.4%), while Taiwan had 40.51% of problems in combination forms.

A study by Hidayah and Forgasz (2020) examined geometry questions in Indonesian and Australian textbooks to investigate the learning opportunities offered to students through these educational resources. The results show that the two textbooks presented more than half of the tasks in multiple representation forms. Furthermore, the Australian textbooks exhibited a higher proportion of combined tasks than the Indonesian textbooks: 69.1% of the Aus\_1 textbook and 73.6% of the Aus\_2 textbook featured combined forms. A similar trend was observed in Indonesian textbooks, where combined representation forms predominated, accounting for 52.8% in textbook Ind\_1 and 66.6% in textbook Ind\_2.

Yang et al. (2017) conducted a comparative study to explore the learning opportunities provided by mathematics textbooks from four countries: Finland, Singapore, Taiwan, and the United States. The study examined the nature of questions included in the tasks and the expectations for students' experiences with each textbook series. The study yielded insights into the differences and similarities across the textbooks, highlighting variations and convergences in the learning opportunities provided by the mathematics textbooks from the four countries. Notably, textbooks used in Finland and Singapore offered a balanced distribution of questions in various forms of representation, enabling students to engage with diverse problem-solving strategies. In contrast, the textbooks used in Taiwan and Singapore relied more heavily on combined representation forms. The survey also discovered that Finnish and American textbooks had more questions addressed verbally (27.9% and 37.1%, respectively) and visually (41.4% and 43.3%). In contrast, Taiwanese and Singaporean textbooks presented more problems in combined form (69.4% and 33.7%, respectively). Interestingly,

mathematical expressions accounted for only 0.4% of the questions in American textbooks. The research findings accentuate the differences in emphasis and approach among mathematics education systems in various countries.

## 2. 1 Theoretical Framework

Representation forms of problems in mathematics refer to the various ways mathematical problems are presented and communicated to students (Mainali, 2021a; Raditya et al., 2020; Zhu & Fan, 2006). According to Yang et al. (2017), representation forms of problems refer to “the diverse methods of expressing, interpreting, and visualising mathematical problems set in textbook tasks. This definition encompasses how mathematical problems are presented, conveyed, and understood through different forms, such as textual, symbolic, graphical, or diagrammatic representations. Winarso (2023) discusses the representation tasks related to decoding, interpreting, translating, and transitioning between these representations. Mathematical representation plays a crucial role in conveying complex concepts into comprehensible language. Effective mathematics instruction integrates various forms of communication and task presentation to cater to diverse learning styles and promote deep understanding (Abed & Al-Absi, 2015). Zhu and Fan's (2006) framework categorises mathematical tasks into four distinct representation types: pure mathematical form, verbal form, visual form and combined form. Tasks presented solely in mathematical notation, using symbols, numbers, and mathematical operations, are considered purely mathematical in form. Mathematical tasks involving pure mathematical form include activities written in pure algebraic notation utilising symbols (Alkhateeb, 2018).

Verbal form tasks are described solely through written language, without mathematical notation or visual aids. Alkhateeb (2018) describes the verbal form of tasks as “tasks that are expressed in text or words only”(p. 137). Tasks presented solely in the verbal form often require students to read and deconstruct the mathematical language into words, including word problems. The verbal representation translates the mathematical language into simple sentences to express problems.

Tasks presented solely through visual aids, such as diagrams, graphs, charts, or geometric figures (Zhu & Fan, 2006), are characteristic of visual form representation. Students are not provided with written descriptions or explanations accompanying the visual representations in this case. Consequently, visual form tasks require students to apply spatial reasoning and visual processing skills to interpret the mathematical information presented. Moreover, students must recognise patterns, relationships, and structures within the visual representations to solve the problems. Ultimately, this type of representation enables students to develop their visual literacy and problem-solving skills in mathematics.

Tasks integrating multiple representation types, such as mathematical notation, written language, and visual aids, are combined form tasks. Zhu and Fan (2006) define a combined form of a task as “the presentation of the problem in a combination of two or more forms of representation” (p. 7). Tasks provided in a combined form include a combination of either text and diagrams/figures or visual and symbolic representations. The definition of Euclidean geometry suggests that most tasks may involve a combination of representation forms.

Zhu and Fan's (2006) task type framework provides a structured approach to analysing and understanding the various ways mathematical tasks are represented, which can inform teaching practices and educational research (Zhu & Fan, 2006).

## 3. Methodology

This study employed the qualitative research approach and adopted a case study design. This study used a textbook content analysis methodology for data collection. A purposive sampling strategy was utilised to select three approved, widely used textbook series that covered Grades 10–12, giving a total of nine textbooks (see Appendix 1). The study purposefully selected Euclidean geometry as the content area for investigation through purposive sampling. Several key factors motivated this selection. Firstly, Euclidean geometry is characterised by its complexity, posing significant student challenges (Chiphambo & Feza, 2020). Moreover, its importance in the CAPS mathematics

curriculum makes it a critical area of investigation. Euclidean geometry is closely related to other mathematical concepts like algebra and measurement. Consequently, students' achievement in Euclidean geometry may have a ripple effect on their overall mathematics achievement. Furthermore, students in Grades 10–12 experience significant learning challenges in Euclidean geometry, as evident in their performance in the Grade 12 end-of-year examination (DBE, 2018). Hence, the selection of Euclidean geometry as the content area for investigation is based on its complexity, importance, relationships with other mathematical concepts, impact on mathematics achievement, and learning challenges.

The unit of analysis for this study consisted of mathematics tasks, specifically those in Euclidean geometry, including both worked examples and exercises. Worked examples provide author-led instruction and modelling procedural and conceptual knowledge, while exercise tasks require student action and application (Ayyildiz & Aktaş, 2022; Firdausi & Jupri, 2020). According to Yilfiz et al. (2020), a task is an activity or question that requires the student's response.

Consistent with existing educational research (Kul et al., 2018), textbook analysis provided a suitable framework for examining curricular materials as empirical evidence. Through this method, the study gained valuable insights into the OTL through representation forms afforded to students by FET phase mathematics textbooks.

This study employed a qualitative deductive content analysis (QDCA) approach to systematically analyse data on OTL Euclidean geometry through various representation forms. Building on existing research, the QDCA method utilised predetermined categories and codes derived from theoretical frameworks and literature (Elo et al., 2014; Elo & Kyngäs, 2008). Each task was evaluated using Zhu and Fan's (2006) type of problem framework, which provided a structured basis for coding and categorisation (See Appendix 2).

Three coders, all master's students in mathematics education, conducted the data analysis. They participated in an initial debriefing session to ensure accuracy and uniformity in coding, building a shared understanding of the theoretical frameworks and material. Each coder individually reanalysed the data to ensure correctness before collectively analysing it to achieve inter-coder consistency. The rigorous methodology of the QDCA approach, including repeated procedures to minimise biases and errors, was further reinforced through methodological triangulation, which compared findings across coders and data sources. These measures enhanced the credibility and trustworthiness of the study.

## 4. Findings

This section of the study presents findings on the representation types of the tasks in the nine mathematics textbooks. The findings are summarised in Table 1.

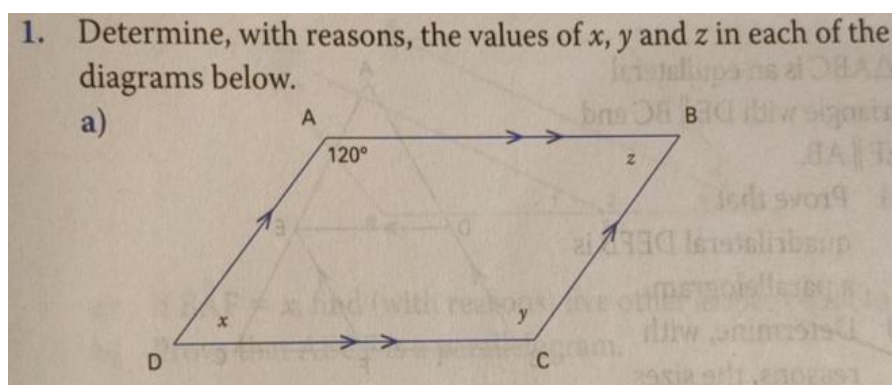
**Table 1.** *The representation forms of tasks in the textbooks*

| Representation Form | Worked examples              | Exercises  | Worked examples             | Exercises | Worked examples    | Exercises  |
|---------------------|------------------------------|------------|-----------------------------|-----------|--------------------|------------|
| Grade 10            |                              |            |                             |           |                    |            |
|                     | Classroom Mathematics (CM)10 |            | Platinum Mathematics (PM)10 |           | Mind Action (MA)10 |            |
| Purely Mathematical | 0 (0%)                       | 0 (0%)     | 0 (0%)                      | 0 (0%)    | 0 (0%)             | 0 (0%)     |
| Verbal              | 0 (0%)                       | 45 (12%)   | 0 (0%)                      | 11 (6%)   | 0 (0%)             | 0 (0%)     |
| Visual              | 0 (0%)                       | 0 (0%)     | 0 (0%)                      | 0 (0%)    | 0 (0%)             | 0 (0%)     |
| Combined            | 22 (100%)                    | 316 (88%)  | 13 (100%)                   | 164 (94%) | 10 (100%)          | 157 (100%) |
| Subtotal            | 22 (6%)                      | 361 (94%)  | 13 (7%)                     | 175 (93%) | 10 (6%)            | 157 (94%)  |
| Grade 11            |                              |            |                             |           |                    |            |
|                     | CM11                         |            | PM11                        |           | MA11               |            |
| Purely Mathematical | 0 (0%)                       | 0 (0%)     | 0 (0%)                      | 0 (0%)    | 0 (0%)             | 0 (0%)     |
| Verbal              | 0 (0%)                       | 0 (0%)     | 0 (0%)                      | 19 (7%)   | 0 (0%)             | 3 (1%)     |
| Visual              | 0 (0%)                       | 0 (0%)     | 0 (0%)                      | 0 (0%)    | 0 (0%)             | 0 (0%)     |
| Combined            | 28 (100%)                    | 249 (100%) | 30 (100%)                   | 243 (93%) | 50 (100%)          | 275 (94%)  |

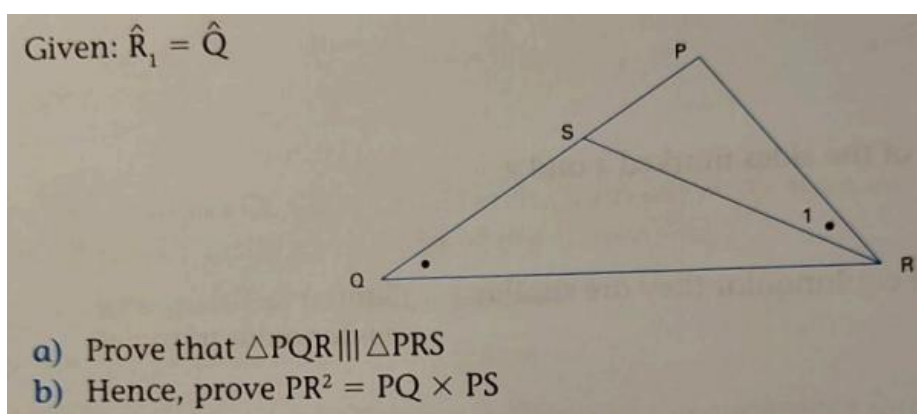
|                     |           |           |           |           |           |            |
|---------------------|-----------|-----------|-----------|-----------|-----------|------------|
| Subtotal            | 28 (10%)  | 249 (90%) | 30 (10%)  | 262 (90%) | 50 (15%)  | 278 (85%)  |
| Grade 12            |           |           |           |           |           |            |
|                     | CM12      |           | PM12      |           | MA12      |            |
| Purely Mathematical | 0 (0%)    | 0 (0%)    | 0 (0%)    | 0 (0%)    | 0 (0%)    | 0 (0%)     |
| Verbal              | 0 (0%)    | 1 (1%)    | 0 (0%)    | 6 (3%)    | 0 (0%)    | 0 (0%)     |
| Visual              | 0 (0%)    | 0 (0%)    | 0 (0%)    | 0 (0%)    | 0 (0%)    | 0 (0%)     |
| Combined            | 20 (100%) | 194 (99%) | 25 (100%) | 190 (97%) | 47 (100%) | 220 (100%) |
| Subtotal            | 30 (11%)  | 195 (89%) | 25 (11%)  | 196 (89%) | 47 (18%)  | 220 (82%)  |

#### 4. 1. Textbook Series 1: Classroom Mathematics Textbooks (CM10, CM11, CM12)

A predominance of combined representation tasks across both worked examples and exercise tasks was found in this series. In CM10, all 22 worked examples and 316 of 383 exercises were in combined form. In CM11, 277 tasks included 28 worked examples and 249 exercises, all of which were in combined form. Similarly, all the 215 tasks in CM12 (20 worked examples and 195 exercises), were presented in combined form. No tasks in any textbook in the series were exclusively mathematical, verbal, or visual. Figure 1 presents some examples of the tasks in the combined representation form in the textbook series.



(a) CM10



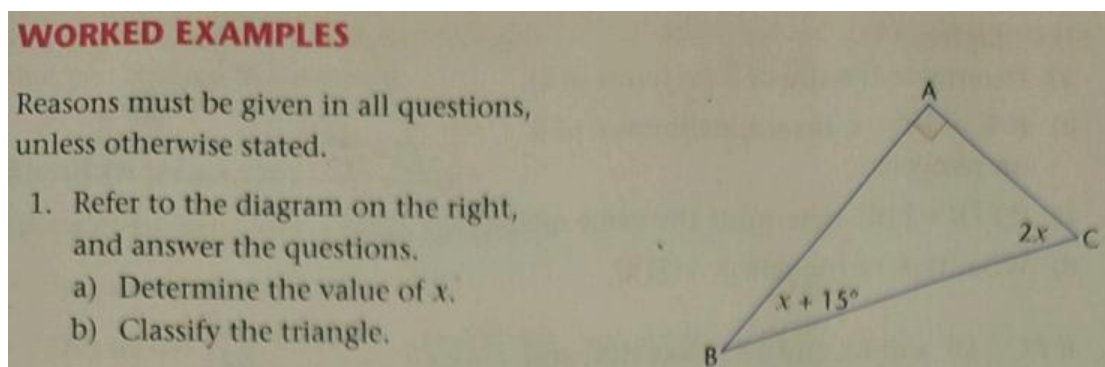
(b) CM12

**Figure 1.** Examples of tasks in combined representation form in the Series 1 textbooks

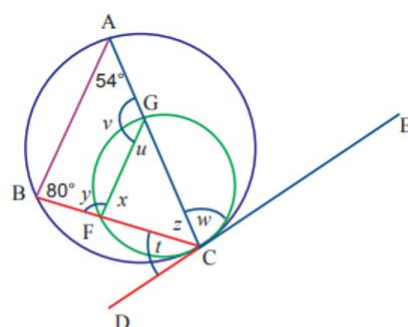
#### 4. 2. Textbook Series 2: Platinum Mathematics Textbooks (PM10, PM11, PM12)

A predominance of combined representation tasks across both worked examples and exercise tasks was found in this series. In PM10, all 13 worked examples and 164 of 175 exercises were in combined form; 11 exercise tasks were in verbal form. In PM11, 292 tasks included 30 worked examples and

262 exercises. All worked examples were in combined form; 243 of the 292 exercise tasks were in combined form, while 19 of these exercise tasks were in verbal form. Similar results were obtained from an analysis of the task in PM12. A total of 221 tasks were analysed, comprising 25 worked examples and 196 exercise tasks. Consistent with the findings in PM11, all 25 worked example tasks in PM12 were in combined form. Furthermore, the analysis of the 196 exercise tasks revealed that 190 were in a combined form, while 6 were in verbal form. No tasks in any textbook in the series were exclusively mathematical or visual. Figure 2 presents examples of tasks in the combined representation form from the textbook series.



(a) PM10



Two circles touch at C. DE is a tangent to both circles at C. A and B lie on the larger circle, F and G lie on the smaller circle.  $\hat{A} = 54^\circ$ ;  $\hat{B} = 80^\circ$ .

- 2.1 Determine, with reasons,  $t, u, v, w, x, y$  and  $z$ .
- 2.2 Is  $DE \parallel BA$ ? Justify your answer.
- 2.3 Is  $BA \parallel FG$ ? Justify your answer.

(b) PM11

Figure 2. Examples of tasks in combined representation form in the Series 2 textbooks

### 4. 3. Textbook Series 3: Mind Action Series Textbooks (MA10, MA11, MA12)

A predominance of combined representation tasks across both worked examples and exercise tasks was found in this series. In MA10, all 10 worked examples and 157 exercises were in combined form. In MA11, 328 tasks included 50 worked examples and 278 exercises; all of the worked examples were in combined form, and 275 of the 278 exercises were in combined form. Similarly, all the 215 tasks in MA12 (47 worked examples and 220 exercises) were in combined form. No tasks in any textbook in the series were exclusively mathematical, verbal, or visual. Figure 3 presents examples of the tasks in the verbal form and combined representation form in the textbook series.

- (g)\* Can you explain why the following statements are true? Draw a sketch if necessary.
- (1) If two opposite sides of a cyclic quadrilateral are equal, then the other two opposite sides are parallel.
  - (2) If a kite is a cyclic quadrilateral, then two of its angles are  $90^\circ$ .

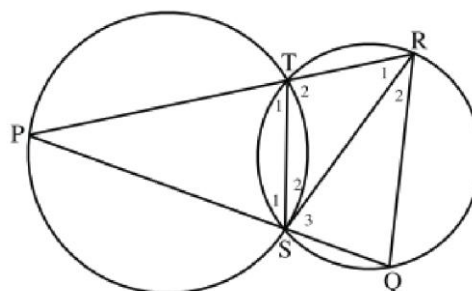
(a) MA11

### EXAMPLE 15

In the sketch alongside, SR is a tangent to circle PST at S.

Prove that

- (a)  $\triangle PRS \sim \triangle SRT$
- (b)  $RS^2 = PR \cdot RT$
- (c)  $\triangle PQR \sim \triangle PTS$
- (d)  $\frac{RT}{PT} = \frac{RS^2}{PQ \cdot PS}$



(b) MA12

Figure 3. Examples of the representation forms of tasks in Series 3 textbooks

## 5. Discussion of the Findings

The findings indicate that the three textbook series overwhelmingly present Euclidean geometry tasks in combined representation forms, with only a small proportion presented in verbal form. Neither purely mathematical nor purely visual tasks were identified in any of the analysed textbooks. These findings suggest that the textbooks provide students with opportunities to learn Euclidean geometry primarily through tasks that integrate diagrams, mathematical notation, and written explanations. Such a pattern is unsurprising given the nature of Euclidean geometry, where diagrams are integral to communicating geometric objects, relationships, and properties. Unlike other areas of mathematics, many geometric tasks cannot be meaningfully interpreted through symbolic notation solely because the spatial relationships among points, lines, angles, and figures are fundamental to the mathematical reasoning required. Consequently, the absence of purely mathematical tasks should not necessarily be interpreted as a limitation of the textbooks but rather as a reflection of the representational demands of Euclidean geometry itself.

The predominance of combined representations is consistent with previous studies reporting that geometry textbooks in several countries rely extensively on integrated visual, symbolic, and verbal representations (Hidayah & Forgasz, 2020; Yang et al., 2017; Yang & Sianturi, 2022). Combined representations are pedagogically advantageous because they connect diagrams with mathematical notation and explanatory text, thereby supporting students' interpretation of geometric relationships and reducing the cognitive demands associated with translating between isolated forms of representation (Rahmatulwahidah & Zubainur, 2017). Similarly, Clements et al. (2018) argue that geometry learning is enhanced when visual representations are integrated with other representational modes, enabling students to relate abstract geometric concepts to concrete spatial structures.

Although the extensive use of combined representations is appropriate for Euclidean geometry, the findings reveal a limited diversity of representation forms across the textbooks. From an Opportunity to Learn perspective, learning opportunities are broadened when students encounter different ways of representing mathematical ideas because each representation emphasises particular aspects of mathematical thinking. Combined representations primarily support the coordination of multiple sources of information, whereas verbal representations encourage students to interpret mathematical language, construct meaning from textual descriptions, and communicate mathematical reasoning (Alkhateeb, 2018; Mainali, 2021). The limited use of verbal-only tasks, therefore, reduces

opportunities for students to develop these important communication and interpretation skills independently of diagrams.

Similarly, although not every geometry task requires a purely visual representation, visual-only tasks can provide valuable opportunities for students to reason directly from geometric figures without relying on accompanying textual or symbolic cues. Such tasks encourage visualisation, spatial reasoning, pattern recognition, and the interpretation of geometric relationships, which are central components of geometric thinking (Clements et al., 2018; Presmeg, 2006). Their absence suggests that students have relatively few opportunities to engage exclusively with visual reasoning, as most diagrams are accompanied by explanatory text or symbolic notation that directs students' thinking.

These findings, therefore, suggest that the issue is not the absence of purely mathematical representations but rather the limited variety of representational experiences provided to students. Opportunity to Learn is strengthened when students engage with multiple representational forms, as different forms support distinct cognitive processes and promote representational flexibility (Duval, 2006; Mainali, 2021; Pedersen et al., 2021). A textbook that relies almost exclusively on combined representations may successfully support integrated reasoning but provides fewer opportunities for students to develop competencies associated with interpreting verbal descriptions or reasoning directly from visual representations. Greater representational diversity would therefore enrich students' opportunities to engage with Euclidean geometry from multiple cognitive perspectives while remaining consistent with the inherently visual nature of the subject.

The findings have important implications for textbook authors, teachers, and curriculum developers. Rather than seeking to increase the number of purely mathematical tasks, textbook authors should strive for a more balanced use of representation forms that is appropriate to the nature of Euclidean geometry. In particular, greater inclusion of carefully designed verbal and visual tasks alongside the dominant combined representations would provide students with richer opportunities to develop mathematical communication, spatial reasoning, and representational flexibility. Teachers can further broaden these opportunities by supplementing textbook activities with tasks that require students to explain geometric ideas verbally, interpret diagrams independently, and translate among different representations. Such instructional practices are likely to promote a more comprehensive understanding of Euclidean geometry and strengthen students' geometric reasoning

## 6. Conclusion

This study investigated how Euclidean geometry tasks in three Further Education and Training (FET) mathematics textbook series provide students with Opportunities to learn through different representation forms. The findings revealed that combined representations overwhelmingly dominated the textbook tasks, while verbal representations were used sparingly. No tasks were presented exclusively in purely mathematical or purely visual forms. The predominance of combined representations reflects the inherently spatial nature of Euclidean geometry, where diagrams, mathematical notation, and written explanations work together to communicate geometric relationships. Consequently, the absence of purely mathematical representations should not be viewed as a weakness of the textbooks, but rather as a characteristic of the design of geometry tasks. However, the findings also reveal limited representational diversity, as students encounter relatively few opportunities to engage with verbal-only or visual-only tasks that foster distinct forms of mathematical reasoning. A broader range of representational forms would expose students to a wider range of cognitive experiences, strengthen representational flexibility, and promote a more comprehensive understanding of Euclidean geometry.

### 6.1. Implications for Students' Learning Opportunities

The findings suggest that the analysed textbooks provide substantial opportunities for students to learn Euclidean geometry through combined representations, which effectively integrate diagrams, symbolic notation, and written explanations to support conceptual understanding. However, the limited use of other representation forms narrows the range of learning opportunities available to students. In particular, the scarcity of verbal-only tasks provides fewer opportunities for students to interpret

mathematical language, construct meaning from textual descriptions, and communicate geometric reasoning independently of diagrams. Similarly, the absence of visual-only tasks limits opportunities for learners to engage directly with figures and develop visualisation, spatial reasoning, and pattern recognition without guidance from accompanying text or symbolic notation. Although combined representations remain appropriate and necessary in geometry, a more balanced distribution of representation forms would broaden students' opportunities to engage with geometric ideas from multiple cognitive perspectives and strengthen their ability to interpret, connect, and translate among different representations. .

## 6.2. Recommendations

Textbook authors should continue to use combined representations as the primary means of presenting Euclidean geometry while incorporating more verbal and carefully designed visual-only tasks to broaden students' learning opportunities. Teachers should supplement textbook activities with tasks that encourage students to explain geometric ideas, interpret diagrams independently, and translate between different representations. Curriculum developers should also consider representational diversity as an important indicator of Opportunity to Learn when evaluating mathematics textbooks, ensuring that students are exposed to a wider range of meaningful representational experiences. .

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## Appendix 1

### Analysed Textbooks

| Publisher            |             | Title of textbook series |        |        |  | Year of Publication | Grade | Code |
|----------------------|-------------|--------------------------|--------|--------|--|---------------------|-------|------|
| Series 1             |             |                          |        |        |  |                     |       |      |
| Pearson South Africa |             | Classroom Mathematics    |        |        |  | 2020                | 10    | CM10 |
|                      |             |                          |        |        |  |                     | 11    | CM11 |
|                      |             |                          |        |        |  |                     | 12    | CM12 |
| Series 2             |             |                          |        |        |  |                     |       |      |
| Pearson South Africa |             | Platinum Mathematics     |        |        |  |                     | 10    | PM10 |
|                      |             |                          |        |        |  |                     | 11    | PM11 |
|                      |             |                          |        |        |  |                     | 12    | PM12 |
| Series 3             |             |                          |        |        |  |                     |       |      |
| All Copy Publishers  | Mathematics | Mind                     | Action | Series |  | 2016                | 10    | MA10 |
|                      |             |                          |        |        |  |                     | 11    | MA11 |
|                      |             |                          |        |        |  |                     | 12    | MA12 |

## Appendix 2

### Representation Form Coding Rubric

| Representation forms of tasks  | Descriptions                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | Code |
|--------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------|
| Purely mathematical (symbolic) | <p>“Representations by written symbols such as writing, numbers or symbols to express the EG concept”</p> <ul style="list-style-type: none"> <li>- Students typically employ mathematical operations and algorithms to arrive at a solution. The self-contained quality of these tasks ensures that students can complete them independently, without relying on extrinsic knowledge or experiences. This type of task requires students to demonstrate their mastery of mathematical concepts and procedures.</li> </ul> | R1   |
| Verbal                         | <p>“Verbal form consists of a task type only described in written words or verbal expressions, which allows for more rephrasing the words and linking the concepts with other areas, and with life situations”</p> <ul style="list-style-type: none"> <li>- Reliance on linguistic descriptions, wherein students are required to interpret and comprehend the mathematical context presented solely through words</li> </ul>                                                                                             | R2   |
| Visual                         | <p>“A visual form comprises figures, pictures, graphs, charts, tables, diagrams, maps, and so forth to express EG concepts”</p> <ul style="list-style-type: none"> <li>- Students engaging with visual form tasks are required to interpret and comprehend the mathematical information presented solely through visual means, without the support of written language or mathematical notation.</li> </ul>                                                                                                               | R3   |
| Combined forms                 | <p>“A combined form is designed with a combination of two or three of the aforementioned forms” (Zhu &amp; Fan, 2006, p. 7)</p> <ul style="list-style-type: none"> <li>- Use a diagram (visual representation), provides written information about the figure (verbal representation), and requires mathematical calculations (pure mathematical representation).</li> </ul>                                                                                                                                              | R4   |