Acta Didactica Napocensia

Volume 16, Number 1, 2023 - DOI: 10.24193/adn.16.1.5

The Roles of Argumentation Structures for The Conviction of Proof Types

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Abstract: This phenomenology research aims to examine prospective elementary mathematics teachers' proving and proof evaluation and their thoughts on convincing according to proof type and argument type. The participants were eight prospective teachers. The data collection tools were semi-structured group interviews, interviews video recordings and the participants' written proof documents. The participants were expected to prove different mathematical statements presented to them with different proof types, to convince each other, and to identify the convincing arguments in the interviews. The results revealed that prospective mathematics teachers had absolute conviction about empirical arguments, while their level of convincing about deductive arguments increased as a result of discussions on convincing regardless of the proof type. In addition, the unconvincing for induction and visual proof types' arguments have emerged and this category has changed to convincing over time. Accordingly, suggestions about increasing the convincing of deductive and visual arguments have been presented.

Key words: Mathematical proof, argumentation structure, convincing, proof type, visual proof.

1. Introduction

In advanced mathematics courses, prospective teachers often encounter proofs. The objectives of these courses include prospective teachers' reading and understanding of the presented proofs and using them as mathematical arguments (Davis & Hersh, 1981). In the process of proof comprehension determining how well these objectives have been achieved, Weber and Mejia-Ramos (2015) emphasize that the learners' evaluation of what the proof is and convincing to this evaluation play key roles for advanced learning. For this reason, they pointed out the importance of the learner evaluating a given mathematical expression as an argument and finding the proof convincing. Thus, the process of deciding the validity of a mathematical argument and evaluating the argument is based on the convincing proof type presented. Proof types can be familiar such as induction and contrapositive from mathematics courses and literature, while some proof type can also be relatively different and unfamiliar such as visual proof. (Bardelle, 2010; Borwein & Jörgenson, 2001; Hanna & Sidoli, 2007). Indeed, the visual presentations contribute to learning such as concretizing abstract concepts, supporting mental manipulation and allowing spatial reasoning (Nelsen, 1993). Therefore, it is important for teaching practices to examine arguments depending on the proof type and the convincing of this type in proving and proof evaluation.

1.1. Mathematical argument, argumentation and proof type

Proof is the sequence of logical and mathematical arguments that clearly and convincingly demonstrate the truth or untruth of the proposition with reasons (Hanna, 2000). Each of these arguments is a necessary step for the proof to be comprehensive and understandable (Duval, 2007; Stefanowicz, 2014). Therefore, while mathematical statement is a comprehensive concept (Weber & Mejia-Ramos, 2015) that specifies precise assertion that have a truth-value, argument is a concept that includes theorems and lemmas that are the products of logical and systematic thinking (Tall, 2008). Weber and Mejia-Ramos (2015) introduced the structures of argument as empirical argument and deductive argument based on this distinction. Accordingly, the empirical argument is that the

Received September 2022.

Cite as: Aktaş, F.N. (2023). The Roles of Argumentation Structures for The Conviction of Proof Types. *Acta Didactica Napocensia*, 16(1), 66-81, https://doi.org/10.24193/adn.16.1.5

correctness of the mathematical expression is valid for finite numbers of the proper subset elements. Deductive argument, on the other hand, is a mathematical statement that consists of a sequence of assertions that are assumed or checked to be true. Empirical arguments can contain deductive assertions based on the axiomatic components of the proposition. Thus, empirical arguments can be components of a conclusive proof that a mathematical statement is true, as well as determine the validity of the claim by checking with examples for a proper subset of all possible cases covered by the claim (Stylianides & Stylianides, 2009).

Garuti et al. (1998) consider argumentation as a process in which the student reasoned individually or with a group to determine the method and steps to be followed to prove a statement. After the argumentation is completed, the process of convincing the individual for the correctness and validity of the decision begins (Weber & Mejia-Ramos, 2015). In convincing process, the justifications for supporting the claim are also accepted as arguments (Pedemonte, 2007). Therefore, proof is a special argumentation process that includes deductive structures as an inference that enables a claim to be made from data and a rule, abductive structures introduced before data is defined or inductive structures as a proof or generalization of special cases (Pedemonte, 2001).

Argumentation is reasoned discourse using arguments and involves the use of all verbal mediators to convince a certain proposition to be true or false (Hanna & de Villiers, 2008). Acceptance of a convincing argument as proof is possible if it is based on known axioms and definitions, proper use of logical notation, and containing proof type signs (Weber & Alcock, 2009). The verified truth of propositions can be demonstrated with inductive and deductive proof types. The main thing here is to choose the proof type proper for the proposition (Moore, 1994). Types of proof by deduction include direct proof, proof by contradiction, and contrapositive types. Induction, moreover, is more advantageous than other proof methods since the argument steps to be followed on the set of natural numbers are invariant (Rossi, 2006). Visual proof, on the other hand, is relatively less familiar than other proof types (Hanna & Sidoli, 2007). Although there are different argumentation structures in the context of examining the proof types (Pedemonte, 2007), the argument structure categories of Weber and Mejia-Ramos (2015) are discussed in the current study to evaluate the arguments conviction.

1.2. Proof and conviction

Proof can be defined as the process by which a person puts forward to remove doubts about whether a claim is true or false (see Harel & Sowder, 1998). Harel and Sowder (1998) categorized the process of removing one's own doubts as "ascertaining" and the process of removing the doubts of others as "persuading". In other words, proof is the process of convincing oneself or another of the truth or falsity of an argumentation (Weber, 2001; 2008). Therefore, an argument that one finds convincing for oneself may not always be proof, as proof also has an aspect that must be convincing when another person reads it. Thus, Selden and Selden (2003) stated that the correctness of a sequence of arguments should be examined for the proof verified. However, the individual's preference for empirical argument to deductive argument or being unconvinced over the deductive argument are not indicators of the individual's inadequacy in proving (Weber & Mejia-Ramos, 2015). These indicators only reflect the individual's thoughts on the correctness or validity of the proof. Indeed, proof is an argumentation that convinces the self or others who have known the content (Davis & Hersh, 1981).

Weber and Mejia-Ramos (2015) discussed the conviction in two dimensions as absolute conviction and relative conviction:

"Absolute conviction to mean that a student, who is convinced of a claim, has absolute certainty that the claim is true. Relative conviction refers to the idea that when a student expresses conviction in a claim, the student is expressing that the probability that they would assign to a claim exceeds a certain threshold" (p.15).

However, the fact that a proof is conviction for an individual is independent of its validity through arguments (Segal, 1999). In fact, the diversity of proof types and the fact that they are based on argumentation lead to difficulties for students -even pre-service teachers- on where to start proof or how to use the conceptual knowledge required in proof (Weber, 2001). In addition, the proof type that convinces the self may differ from the types that convince others depending on the concept knowledge (Pfeiffer, 2011; Weber, 2010). Therefore, the difficulties of prospective teachers in determining whether a given proof is true explains their relative conviction that the proof validity. Indeed, conviction is a prerequisite for proof (Weber & Mejia-Ramos, 2015). Therefore, it is important to know the learners' ideas about proof and argumentation, the proof types and arguments that convince them to use proof effectively. Thus, it will be possible for prospective teachers to be convinced of the proof importance as the learners for the learning mathematical concepts and to have an idea about convincing others -especially their students- as future teachers in educational practices.

Prospective mathematical teachers' conviction processes (Weber, 2010), their attitudes towards technology-supported proof strategies (Inglis & Mejia-Ramos, 2009; Moralı, Uğurel, Türnüklü & Yeşildere, 2006; Stylianides & Stylianides 2009; Zengin, 2017) and their experiences (Demircioğlu, Examinations with a single perspective such as 2019; Knuth, 2002), proof evaluation (Doğan, 2020; Inglis & Aberdein, 2015; Pfeiffer, 2011), or validity (Recio & Godino, 2001; Selden & Selden, 2014) are draw attention. In this literature, there are also suggestions for examining convincing processes by conducting in-depth research based on proof types (Inglis & Mejia-Ramos, 2009; Stylianides & Stylianides 2009). The current research will contribute to the literature on how advanced mathematics arguments should be presented to prospective teachers, and the choosing or variety of proof types. Thus, the aim of this study is to determine the role of arguments and argumentation according to the proof types so that prospective mathematics teachers are convinced of the correctness and validity of proof in the reasoning process. Accordingly, the research problem is as follows: What is the role of the argumentation structure according to the proof types in convincing prospective mathematics teachers about the correctness and validity of proof and convincing their peers?

2. Method

2. 1. Research design and participants

The research was designed in phenomenology, as it examines the convincing processes of prospective teachers by exploring their thoughts (Merriam, 2009) based on their experience of proving and examining. The participants of the research were eight prospective elementary mathematics teachers. According to the criterion sampling method, the participants were senior students with different academic achievement standings, who had taken the courses for content knowledge and pedagogical content knowledge, particularly the Logical Reasoning elective course. The participants with the code names Mete, Furkan, Cenk, Okan are male and Burcu, Ceyda, Ece, Melek are female.

2.2. Data collection tools and procedure

Data collection tools were interviews via group discussion, video recordings of these interviews and written documents of participants' proofs. It was possible to increase reasoning and convincing through group work (see Garuti et al., 1998; Weber & Mejia-Ramos, 2015). Indeed, it is known that proof evaluation is more effective with small groups and discussing (Weber, Maher, Powell & Lee, 2008). In three sessions, different mathematical statements were presented to the participants and they were expected to prove and convince each other with different types of proofs. The turn of the participants to explain their proofs based on types was left to their choosing. Then, the participants were asked to indicate the arguments that convinced them for each proof types. In the first two sessions, it was requested to examine at least three different proof types -at least one of which was visual proof- for each of the mathematical expressions. Thus, the participants also identified the proof type that convinced them the most. In the last session, four different visual proofs for the Pythagorean theorem were presented. The mathematical statements presented are "if n is an even integer, then 3n+7 is an odd integer", "if n is a positive integer, then n^2+n is an even integer", and the Pythagorean Theorem.

The expert opinions were obtained from mathematics and mathematics education professors for the designed interview protocols. Accordingly, verbal expressions were added to some visual proofs. Then, a pilot study was conducted with a prospective elementary mathematics teacher. Accordingly, a different visual proof with variables was added for the Pythagorean theorem.

2.3. Data analysis

Data were analysed by content analysis method. Data from participants were transcribed, then, the raw data read were categorised with reflective notes in different contexts on proof types, argumentation, arguments and convincing. Accordingly, the categories in the results as in Figure 1 were obtained.

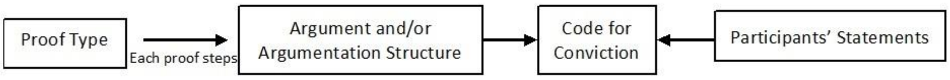

Figure 1. *Data analysis process*

2.4. Validity and reliability

By determining the participants with the purposeful sampling method, the transferability of the results to other environments has been increased. In addition, getting expert opinion for data collection tools and conducting a pilot study contributed to increasing the reliability of the research. The data were coded by the second encoder and a consistency study was conducted for the categories. Accordingly, by making changes on the codes of conviction and argument type, different convincing types among the participants were considered as separate categories for the same argument types.

3. Result

3.1. The convincing argumentation structures and proof types in first session

Participants tried to prove the proposition "if n is an even integer, then 3n+7 is an odd integer". They initially tried to prove it with empirical arguments and stated that they had *absolute convincing* with finite number of example cases:

Mete: We know that 2 is an even integer. Then, the odd number 13 is obtained from the statement.

Melek: I generally use an even number instead of n at 3n+7. If it's an even number, it's already contradictory proof. I think it's enough.

Mete: Hmm… if not?

Melek: I try a few more examples, but I don't need it.

After the idea of *generalization* emerged, they continued to argue with *direct proof* and *contrapositive proof*:

Ece: It should be generalizable. Let's use 2a instead of n. So, we will get an even number. Because n is an even integer. Then we use 2a instead of n from 3n+7 in the result. 6a+7 is an odd number.

Okan: You're right. In fact, I also thought the contrapositive proof. If 3n+7 is even, then n is even. Or for the assumption of n is even, we can use properties such as multiplication of an even and odd numbers.

Convincing arguments were discussed by proving according to the proof types suggested:

Ceyda: For direct proof, wouldn't we accept 2k as Ece said? There is an integer k, such that n=2k.

Ece: We write 2k instead of n at 3n+7. The result is a statement like 2m +1, an odd integer.

Cenk: The last step convinced me that we get 2m+1 odd integer since it is $k \in \mathbb{Z}$. The hypothesis about n was also important in convincing me for it to be valid.

Mete: We just said that when we write certain numbers at statement, there may be elements that are not correct. This generalization with 2k convinced me.

In the argumentation for *direct proof*, the mathematical statements for *hypothesis*, *generalization* and *conclusion arguments* are convincing criteria. Here, *deductive arguments* include *the definition of even integer* and *mathematical operations*.

Furkan: Ok. In the contrapositive proof, we assume that there is 3n+7 even number. Then n is odd. But at first n was an even number. Thus, acceptance and hypothesis contradict. [He completed proof]

Melek: I was convinced in the third step because here the proof is complete [There is an integer y-n-4 such that n=(3n+7)-(2n+7)=2y-2n-7=2(y-n-4)+1 is the mathematical expression]. I am more convinced of the direct proof. This proof is complicated.

Burcu: Since we can see that the expression we obtained as 3n+7=2y is even, this is the most important step. This is the step that confused you.

Okan: Actually, first I thought the contrapositive proof. I couldn't express the logically equivalent contrapositive statement. But, for direct proof, I think it is more difficult to remember the lemmas and theorems and also complete the proof.

Cenk: Direct proof steps come to mind immediately, so it's easy and convincing. But the concluding step convinced me of the correctness of this contrapositive proof. Because we got the result of contradiction there.

The *contrapositive proof* is a *not clear* proof as it prompts participants to think more about expressing through *propositions* and writing in *mathematical expressions*. However, when the participants examine their peers' proofs, they have *absolute convection* for *deductive arguments*, which are reached *contradiction* and *initiation*. The main thing for both arguments is argumentation by writing it as a *mathematical expression*. Indeed, participants confuse what is given and required as true or false for *contrapositive proof*. When the researcher proposes the idea of visual proof:

Cenk: Could it be a shape pattern? A pattern like 2n+2 after 2n.

Ece: There cannot be a clear proof, but something can be done to help prove it. Maybe an argument.

Burcu: Modelling. For example, something like area model can be done.

Furkan: The units like fractions can be used. We can combine two units for even integer and use an unit for odd integer.

Okan: It can also draw a schematic. For n even numbers, using a two-part figure for 2n initially. For 3n+7, it can be 3 of the same models plus 7 [They proved as Figure 2].

Burcu: The proof is convincing because it is different from the others. The last step is more convincing, it's clear what's going on. Other proofs are full of more academic writings. But this is more visual. Like a modelling method to aid proof.

Melek: This is the most illogical of the types of proofs we've seen. Expressing an integer with a shape is very illogical.

Furkan: But we can show this proof to students according to grade level. We represent the numbers with beads.

Cenk: I am absolutely very convinced and it is absolutely true. This was a bit strange since we always see numbers. The numbers are more convincing. If there was an explanation, I desperately needed it. For that I would do the contrapositive proof.

Mete: In fact, these goes step-by-step like the others. But it's more important to think about shapes in the first step.

Ece: This proof convinced me the most. More concrete is what I envision in my mind. If I was a teacher, I would use this. For example, I thought of a basket, I put 3 letters n and 7 balls. I grouped in threes. Since 2 squares will be even numbers in every time.

Furkan: But it is n=2. I think this is a demonstration, not a proof. I was not convinced. Because there are no mathematical operations.

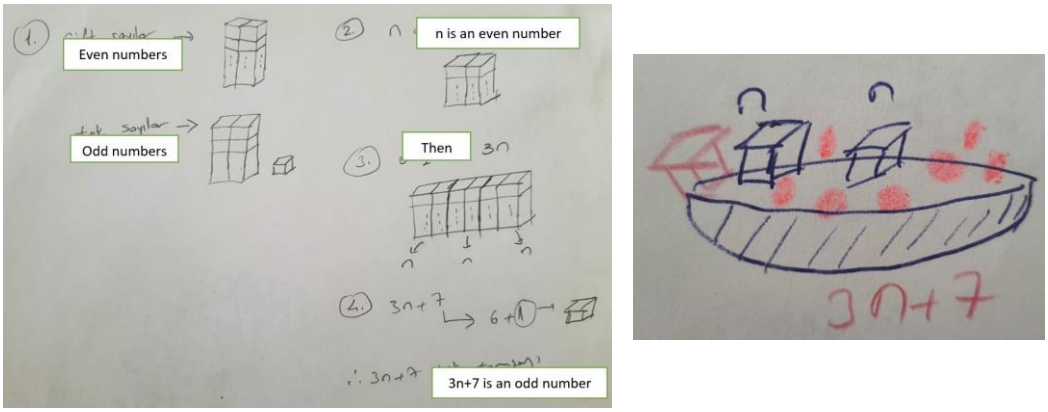

Figure 2. *The participants' visual proofs*

Participants who encountered *visual proof* in the Logical Reasoning course still explained visual arguments as *modelling*, *concrete representation*, *pattern*, and *schema*. Therefore, participants had *relative convection* for visual proof. In fact, it was *unconvincing* for Furkan. Participants wanted to prefer visual proof according to the individuals they would convincing. They also tried to make different visual proofs for middle school students [see Figure 2]. Furkan, who thought the contrapositive proof more convincing, was convinced of the visual proof at the last argument. Similarly, Cenk, who convinced to direct proof, was convinced at the last argument to both the contrapositive proof and visual proof. Mete, who more convincing to direct proof, focused on the convincing of the first argument for visual proof as well as direct proof. It was determined that the participants pay attention to *deductive arguments*, needed *mathematical expressions* and *equations*, and focused on *the first and last steps* of the proof for convincing. The results are summarized in Table 1.

Table 1. *The Convincing Argumentation Structures and Proof Types in First Session*

Proof Type	Proof Step	Argument	Conviction
Not argumentation	An example for the mathematical statement	Empirical argument	Absolute conviction
Direct proof	Hypothesis statement	Deductive arguments	Absolute conviction
	Generalization		
	Conclusion statement		
Contrapositive proof	q' hypothesis statement	Deductive argument	Absolute conviction
	Mathematical statement for process		Relative conviction – not clear proof
	Conclusion statement		Absolute conviction
Visual proof	Hypothesis statement	Deductive argument	Relative conviction – not proof
			Absolute conviction
	Mathematical statement for	Deductive	Unconvincing

	process	argument	
	Conclusion statement	Deductive argument	Relative conviction – not proof
			Absolute conviction
Induction	Not argumentation – proof not completed	-	-
Contradictory proof	Not argumentation – proof not completed	-	-

3.2. The convincing argumentation structures and proof types in second session

Participants were expected to prove the proposition “if n is a positive integer, then n^2+n is an even integer”, determined proof types and convinced each other.

Mete: For $n \in \mathbb{Z}$, n(n+1) by parenthesis n, so one more. When we multiply, we get an even integer. It’s like 4 times 5. Multiplication of consecutive numbers.

Ece: Yep. I normally like the contrapositive proof, but now direct proof is obvious. It makes sense to bracket and then to value as Mete did.

Furkan: But we're not proving it. I thought of visual proof, but I don't know how to represent n^2.

Mete: Do we need to assume that n is even or odd?

Melek: We will look at two cases being even and odd.

Ece: It must be induction. Shall we start with even or odd?

Melek: Then the situation becomes proof by cases. In induction, we have to examine for 1, 2, 3, 4 and then write the general expression.

Ceyda: Or it is necessary to show that the negation does not provide for the general expression. We cannot try one by one for infinite numbers. If it does not provide the negation situation, ok.

Furkan: We will just give an example again. Let's examine for even and odd then.

Participants first discussed *inductive arguments* through *examining cases*. They then decided on *proof by cases* through these arguments. Here, the convincing criterion in the choosing of proof type is *generalization*. They preferred proof by cases as they focused on *examples* and *concept definitions* as empirical arguments. Then they examined Furkan’s idea of visual proof:

Mete: Let n be one side of the square. If we consider a rectangle, its short side is n and its long side is n+1. For n brackets in area formula, ok.

Furkan: Let’s consider a fraction bar because 1 unit is like n units.

Burcu: We can form rectangles with small squares [wrote the proof in Figure 3]. We will represent rectangle for n^2.

After Burcu completed the visual proof, they discussed convincing:

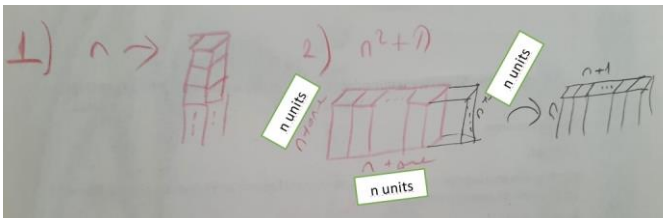

Figure 3. *The participants’ visual proof*

Ece: I couldn't even think of the first step. But the second step convinced me more. In fact, we were familiar with.

Melek: It’s really convincing, since in the last step a rectangle with the side n and other side n+1 is formed. Without leaving a question mark!

Ceyda: It’s like modelling factorization actually.

Burcu: I am convinced that all the steps are related as a whole. But the proof takes shape in the second step.

Familiarity and *argumentation* are codes of convincing for *visual proof*. Participants reasoned for a long time for *induction*. They tried to *induction* with empirical arguments. They have *relative conviction* about the assuming of the *p(k) truth value* and have tried to benefit from *visual proof*.

Furkan: We start for n=1. We will examine the truth values for p(1) and p(2).

Ceyda: No, after examining for p(1) we will reason for n=k.

Okan: Let it be true for claim k, should we not write it?

Ceyda: How do we write then? Without assuming, do we get the proposition in the second step that if k is odd, then k+1 is even?

Mete: But this also applies to the initial proposition. We already assume k for generalization in induction.

Furkan: How do we verify that k is true?

Cenk: Let’s do it as in the visual proof.

Okan: But then would you have induction?

Furkan: After assuming k=1, we need to verify that it is true for n=k+1. We will show that this is even number.

Burcu: What if it wasn't true for n=1?

Ceyda: Proof ends if it's not true for one. [They completed induction]

Furkan: The arguments that convinced me the most are “what if k=2 if true for p(1)?”. This time k=2 will be correct. Accordingly, k=3 will be correct. This is how it goes forever.

Burcu: Exactly. We accepted that it was true for the claim. I said can I be convinced there? But then it convinced me. I said ok we did the induction.

Cenk: We already knew that p(k) from the expression k+1 in parenthesis of 2 was even. Here I am convinced. Already the result is even number.

Participants questioned the *induction* steps and therefore they had relative conviction for *hypothesis step* and *inductive step* by focusing on *basis step*. Participants who even thought of making visual proofs due to induction difficulties discussed proof types in the context of conviction:

Ece: I was most satisfied with proof by cases. We were not familiar with visual proof, either we did not know induction. The first one in my mind was proof by cases.

Furkan: It was easier because it started from the definition.

Burcu: It seems more logical here, but induction is more useful when the number of cases is more than two.

In Table 2, participants’ proof types and convincing types obtained in the second session are presented.

Table 2. *The Convincing Argumentation Structures and Proof Types in Second Session*

Proof Type	Proof Step	Argument	Conviction
Not argumentation	An example for the mathematical statement	Empirical argument	Absolute conviction
Proof by cases	Hypothesis statements	Deductive argument	Absolute conviction

Visual proof	Hypothesis statement	Deductive argument	Relative conviction
	Mathematical statement for process		Absolute conviction
	Conclusion statement		Absolute conviction
Induction	Basis statement	Empirical argument	Absolute conviction
	Hypothesis statement	Deductive argument	Relative conviction
	Inductive statement		Relative conviction

3.3. The convincing argumentation structures and proof types in third session

After increasing familiarity with visual proof, different visual proofs of the Pythagorean Theorem were examined for a detailed examining of convincing. Indeed, in the sessions, the participants did not make different visual proofs and did not use the priority of preference in favour of visual proof. Therefore, visual proofs of the Pythagorean Theorem, familiar to the participants, were included. However, initially participants were expected to prove with whatever proof types they choose:

Ceyda: Visual proof, right? The sides of the triangle and the square have common sides.

Cenk: We can also prove it with the cosine theorem. Doesn't have to be visual. $\cos 90^{\circ}=0$.

Burcu: But we have to use Pythagorean Theorem to prove the cosine theorem.

Ceyda: Let's draw three separate squares. Its sides are a, b, and c. To form a triangle. Let's prove it using areas.

Ece: I think we should use visual proof. We cannot prove step by step in geometry. For example, I can take advantage of their area by drawing another square inside a square.

Mete: We may need to take advantage of the similarity for triangle.

Ceyda's idea of visual proof was presented [see Figure 4] since the participants could not complete other proofs:

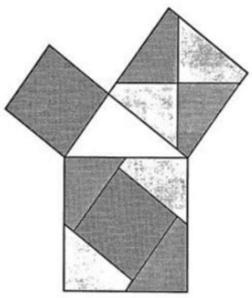

Figure 4. *Visual proof for Pythagorean Theorem*

Cenk: I think it proves with areas. Assuming the areas as a^2, b^2 and c^2, we have a^2 and b^2 for the big square.

Burcu: The side lengths are not clear. Equality of areas is evident when we carry it like a tangram.

Mete: True if they are all squares but not possible if they are not squares.

Furkan: Exactly, for example, can we do this covering no matter how we divide it here? Looks like some rules are needed in shapes too.

Ceyda: I was not visually convinced either. Does it fit exactly there? If I had it as a concrete material, maybe I would be more convinced. I wouldn't be convinced if I didn't know the truth of the theorem.

Participants needed *explanations* about the figure and *mathematical symbols* for the precision of the visual proof in Figure 4. They also emphasized that *concrete materials* are needed to be convinced of the proof. Participants had *relative conviction* and *unconvincing* for the proof with *empirical arguments*. Due to the emphasis on *symbolic language* and *mathematical operations*, the proof in Figure 5 was presented to the participants:

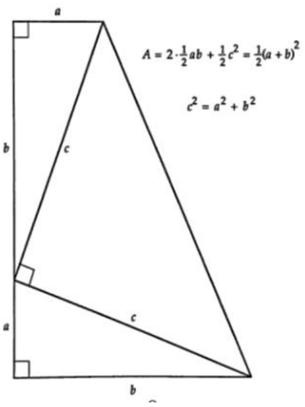

Figure 5. *Second visual proof for Pythagorean Theorem*

Cenk: The sum of the areas of the triangles is equal to the area of the trapezoid. I'm convinced because it uses variables. I think the previous proof might be more convincing because of square areas, clear!

Mete: This is more convincing, I think. The arguments are clear here. The sum of the areas of the three triangles and the area of the trapezoid. The fact that there are operations is more persuasive. The areas in the previous proof are not clear.

Ece: If we don't know that it is a square in the previous proof, do we know the parallelism of the trapezoid here?

Furkan: It indicated the vertical.

Ece: Ok I was convinced (smiling)

Furkan: In the previous proof, we could operate by giving variables such as x and y. Then it might be more convincing.

Burcu: We need to write a variable and validate it. Both convinced, but the previous proof is more convincing because it's something I've seen and become familiar with. In Material Design course, there was a material for the squares where we put water. Also, as if the argument would be more correct when it was operational.

Ece: I wonder if it started from the trapezoid or triangle?

Mete: As if to consider the area of the trapezoid at first, and then the sum of the areas of the triangles. More convincing than the other because it is obvious that we use arguments and operations are.

Furkan: Exactly! We can take advantage of both the similarity and the area formulas. The other is just visual, it is more difficult to relate it.

Since the participants thought that the arguments for visual proof were not clear, they tried to identify the arguments with *operations* or *symbolic representations*. They also obtained deductive arguments such as *area* and *similarity* through operations. Therefore, they identified the proof in Figure 5 as more convincing. The proof in Figure 6 was presented to the participants who thought that their convincing would increase by specifying the variables in the first proof:

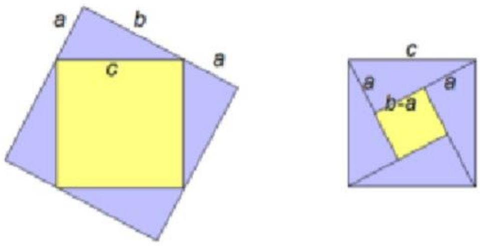

Figure 6. *Proof with variable for Pythagorean Theorem*

Furkan: They folded the triangles inside. This long-folded edge and the short edge are in the same line. So, b-a. Then?

Ece: First we calculate the area of the square. What if we calculate the right little square?

Okan: We should start from the left square.

Ece: Then $(a+b)^2$ is equal to the sum of the areas of the right triangles. One minute. Equals 4ab/2. The area of the square inside is c^2. $a^2 + b^2 = c^2$.

Melek: It should show operations. I don't think it's clear.

Ece: At first, we did not know whether it was left or right. If there were operations, we would know which side to use. However, in the first proof, the areas of the triangles could be interpreted.

Burcu: We call them concrete, but area and similarity are more familiar. So, the second proof is more convincing.

Although there are *variables* in the visual proof, the participants argued that having *operations* would be more explanatory and convincing. They were more convinced of the validity of the first two visual proofs. The proof in Figure 7 was presented as the participants were more convinced of the *area* and *similarity* arguments:

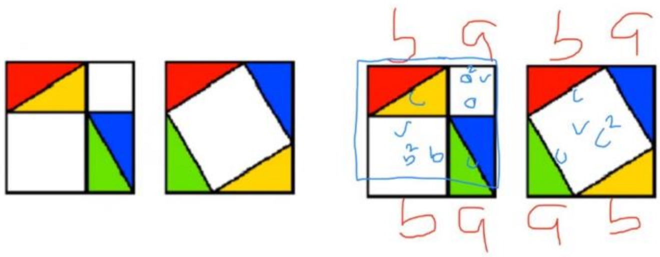

Figure 7. *Visual proof for Pythagorean Theorem*

Cenk: Looks like the first proof. For example, the sum of the red, yellow, blue, green areas is 2ab, and the areas of the other squares are a^2 and b^2. When we place it like the square on the right, for example, since 2ab is the perimeter, a^2+b^2 is equal to the square in the middle. Since the area of the middle square is c^2. $a^2+b^2 = c^2$.

Burcu: When he did the operation, the relationship between the sides of the right triangle emerged. How did we become convinced of the first proof? (smiling).

Cenk: We were convinced of the first proof because we thought of it like a jigsaw puzzle with seeing. But here a little more complicated operation is required.

> Ceyda: I think it is more convincing when we explain it with operations. Since we saw the first proof before, we immediately believed it to be true, but if we showed it to a student for the first time, it might be meaningless.
>
> Melek: Yep. I was also convinced of the first proof, but now I think that the proof was not enough. Subsequent proofs made me think that operations must also occur. We have difficulty in explaining, we have to make use of variables.

Participants think that the use of *concrete materials*, *variables* or *different colours* in the image is not enough to convince the proof of its correctness and validity, and that *mathematical operations* or *explanation* should be included. However, according to the participants, even if these operations or explanations are not included in the proof visually, they should be clearly visible from the visual. Participants also noted the importance of *familiarity* in convincing, noting that convincing will change as the *diversity of proof types* for proposition and the *frequency of encounters with proof types* increase. The argument and convincing types for visual proof are presented in Table 3.

Table 3. *The Convincing Argumentation Structures and Proof Types in Third Session*

Proof Type	Proof Step	Argument	Conviction
Visual proof	Just the figure(s)	Empirical argument	Unconvincing
	The coloured figure(s)	Empirical argument	Relative conviction
			Unconvincing
	The figure(s) with concrete material(s)	Empirical argument	Relative conviction
			Unconvincing
	The figure(s) with symbolic mathematical statement(s) (e.g., variables)	Deductive argument	Relative conviction
			Absolute conviction
	The figure(s) with the mathematical explanations (e.g., mathematical operations, area or similarity concepts)	Deductive arguments	Relative conviction
			Absolute conviction
Algebraic proof	Not argumentation – proof not completed	-	-

4. Conclusion and Discussion

In addition to the absolute conviction and relative conviction components of the proof convince self or others in convincing perspective of Weber and Mejia-Ramos (2015), unconvincing has emerged in the current study. The unconvincing proof is based on evaluation criteria such as the lack of clarity or the absence of familiar proof steps for mathematics prospective teachers. Accordingly, convincing proof is related to proof types and to being the proof based on empirical or deductive arguments. Indeed, while mathematics prospective teachers had the absolute convincing for empirical arguments such as giving examples or trial and error with concrete materials at the beginning of the proving and proof evaluation interviews, they became convinced of proofs based on deductive arguments over time. Depending on this change, the participants' convincing types and preferences have evolved from examining cases to proof with sequences of logical propositions. One of the important factors creating this change is the effort to convince peers of the correctness or validity of the proof in group discussion (Weber et al., 2008).

It is known that students are convinced of the validity and correctness of the proof via empirical arguments (Recio & Godino, 2001). However, some researches emphasize that prospective teachers are convinced by empirical arguments to convince others (Inglis & Mejia-Ramos, 2009; Stylianides & Stylianides, 2009; Weber, 2010). For example; prospective teachers may prefer visual proof based on empirical arguments to convince elementary school students according to the proof content. Here, conviction someone can be thought of as a result of the proof validation process (see Segal, 1999; Selden & Selden, 2003; Weber, 2008). Therefore, conviction someone is not the distinguishing variable for convincing proof. Indeed, convincing begins with conviction self firstly (Weber, 2001). However, this situation is not reflected in having absolute conviction or relative conviction for the proof according to the proof type. Hence, regardless of the type of proof, participants had absolute conviction for deductive arguments in the hypothesis, mathematical operations or logical equivalences as mathematical statements, and contradictions in the conclusion steps of the proof, or statements based on the mathematical definitions. Therefore, arguments based on mathematical statements that contain operations, definitions, symbolic representations, or properties are more conviction. These preferences apply to types of proofs based on deductive arguments, such as direct proof and contrapositive proof, as well as visual proof. For example; in the argumentation process of direct proof, mathematical statements are convincing criteria for hypothesis, generalization and conclusion arguments. Therefore, the types of proofs made with mathematical statements that mathematics prospective teachers are familiar with are more convincing. Indeed, mathematical statements are true claims and convincing (Weber & Mejia-Ramos, 2015).

Prospective teachers described the visual proof with terms such as schema, model, figure pattern. The main reasons for these perceptions are that prospective teachers are not familiar with this proof type (see Hanna & Sidoli, 2007), they think that it is not based on argumentation, and it is no clarity (see Inglis & Mejia-Ramos, 2009). Therefore, in the first group discussion, there were participants who were not convinced by visual proof, participants who thought it might be a suitable proof type for students, and participants have absolute conviction. However, in the following sessions, the convincing level of the participants increased as their familiarity and their ability to identify arguments for visual proof increased. In addition, although the participants were immediately convinced of the first proof presented for the Pythagorean theorem, they changed their thinking for the further proofs in the group discussion and associated this situation with familiarity. Indeed, visual proof is not in the form of numbers, variables, or sequences of operations, unlike the types of algebraic proofs that participants are familiar with (Hanna & Sidoli, 2007). Therefore, participants evaluate the convincing of visual proof according to the proof type they are familiar and convincing. In fact, some of the prospective teachers who frequently apply the proof type that the concluding argument is important, such as contrapositive proof, were convinced by focusing on the last argument of the visual proof, and some of them who prefer direct proof that the determining of the first hypothesis argument is important were convinced by focusing on the initial argument of the visual proof. In addition, the components that make up the unconvincing category focus on the reasons that do not convince the prospective teachers (such as not familiar and lack of proof steps) and the qualities of visual proof (such as without words and visual form). These components were discussed by Borwein and Jörgenson (2001) as reliability, consistency and repeatability. Thus, the visual proof in first engagement practices may need the explanations or argumentation using steps. Therefore, prospective teachers prefer other proof types instead of visual proofs that they are not familiar with and cannot determine the proof steps (see Bardelle, 2010). According to these results, while presenting proof to the prospective teachers, including different proof types in different order and for each type are strengthen convincing and interpretation.

The arguments, such as the supporting with concrete materials and colours, using variables, adding explanatory or clue algebraic expressions (see Hanna & de Villiers, 2008; Inglis & Mejia-Ramos, 2009), adding symbolic language or operations (see Pedemonte, 2007), should be included or available from the visual proof for the absolute conviction. Indeed, participants had difficulties in determining the first hypothesis, determining the arguments, maintaining or concluding the proof for visual proof, and had relative conviction or unconvincing about visual proofs presented. On the other hand, it is an important result that participants having relative convictions are suspicious of visual elements. For example, participants argued that geometric objects such as squares and trapezoids should be specified in the proof or that there should be markers (e.g., perpendicular and side length) proper for concept definitions. Thus, all the steps in the proof should be necessary (Duval, 2007; Weber & Alcock, 2009) and every step of the proof should be made sense in the mind (Duval, 2007). Therefore, the proof should have qualities such as clarity, sufficiency without excess, insight, convincingness or enhancement of understanding (Inglis & Aberdein, 2015; Pfeiffer, 2011).

The prospective mathematics teachers have difficulties in writing logical propositions and determining hypotheses for proof by contradiction. Similarly, they had not clear with the induction steps and confused it with proof by cases. In addition, the participants generalized the result of an integer in the inductive basis step and perceived an case for proof as argumentation. Therefore, they primarily preferred proof by cases through inductive arguments as a result of group discussion. In addition, the criteria for the convincing of proof by cases, such as the arguments based on concept definitions and being the first proof type that comes to mind, are the results of inductive arguments (Pedemonte, 2001; Stylianides & Stylianides, 2009). Therefore, one of the arguments convincing the prospective mathematics teachers is generalization. However, prospective mathematics teachers could not understand the reason for assuming the truth value of p(k) as 1 for induction, so they had difficulty in convincing induction. This result is already among the difficulties that prospective teachers have for induction in the literature (see Stylianides & Stylianides, 2009). The current research has shown that they are more convinced of the less familiar visual proof rather than induction. However, the visual proof was not the first choice of prospective teachers in any of the sessions. Even though the proof of the Pythagorean Theorem including visual elements was the visual proof that came to mind, the participants tried to make the algebraic proof firstly. This preference is a result of the participants' thought that they could not obtain deductive arguments for visual proof. Indeed, participants lacked the ability to consider the first step in designing a visual proof and identify arguments in the visual proof. This result may be due to the difficulty of choosing the proof type and starting the proof (see Rossi, 2006). On the other hand, being unfamiliar with the proof types points out that the reason for the difficulties of prospective teachers in making proofs is the lack of strategic knowledge (Weber, 2001). For this reason, it is important to provide prospective teachers with effective strategic knowledge and to engage with different proof types. Therefore, it is necessary to design educational opportunities for prospective mathematics teachers to prove with different proof types, discuss them, convince the self and others (Pedemonte, 2007; Weber, 2004).

5. Limitation and Further Research

As a limitation, the current research examines conviction processes in the context of doing proof and proof evaluation. Although the results reveal the relationships between proof validity, proof construction, proof comprehension and proof conviction, the convince processes under these contexts should be examined in detail in future research. Thus, detailed examinations of proof conviction will be possible and educational practices can be organized accordingly.

In order to determine the conviction of proof and proof types, different proof types were limited to proofs that prospective mathematics teachers completed in research sessions. In future research, the roles of different proof types, such as proof by contradiction or algebraic proof, which were not revealed or completed in the current research, and the arguments convincing for these proofs can be determined.

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