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FROM PROCEDURES TO STRUCTURE: WORD PROBLEMS AS MARKERS OF PUPILS' ALGEBRAIC READINESS

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Abstract: The transition from arithmetic to algebra represents a critical developmental stage in pupils' mathematical thinking. This study investigates the extent to which Grade 7 pupils who have not yet received formal instruction in algebra can apply procedural strategies rooted in arithmetic and whether they exhibit emerging structural thinking. This study investigates the arithmetic reasoning and problem-solving strategies of Grade 7 pupils when solving word-problems using almost exclusively arithmetic methods taught in earlier grades. A group of 20 pupils completed a set of nine word-problems designed to distinguish between procedural (operational) and structural (relational) modes of thinking. The tasks varied in their underlying mathematical models (e.g., additive vs. multiplicative relations, implicit algebraic structures), allowing for a detailed analysis of pupils' conceptual understanding. The study additionally examines problems in which the unknown variable occurs on each side of the equation, evaluating students' capacity to represent and solve these problems without prior formal algebra instruction. Results show that while most pupils successfully applied arithmetic procedures, a smaller proportion demonstrated structural thinking sufficient for the transition to algebraic methods. Therefore, findings suggest that while procedural thinking is dominant, some pupils demonstrate early structural thinking, particularly on tasks of higher relational complexity. These insights highlight the need for deliberate instructional design to support the gradual shift from procedural to structural mathematical thinking in lower secondary education.

Key words: arithmetical method, algebraic method, operational (procedural) thinking, structural thinking, cognitive abilities.

1. Introduction

The shift from arithmetic to algebra has been identified as a particularly demanding intellectual change for pupils learning school mathematics (Kieran, 1992). While arithmetic emphasizes the application of concrete numerical procedures, algebra requires pupils to conceptualize numbers and operations as abstract objects, symbols, and relations. This shift involves a profound change in cognitive orientation: moving from a *procedural thinking*, where success depends on carrying out step-by-step operations, to a *structural thinking*, where pupils must perceive, represent, and manipulate underlying relationships (Sfard, 1991). Many word problems can be successfully approached by arithmetical strategies, and their solution process requires only a procedural way of

thinking. For instance, problems that can be modelled by a simple equation of the form $Ax + B = C$ such as "a number increased by 12 equals 45") can be handled easily working backwards. So, such problems also allow the use of sequential arithmetic or visualization strategies, such as bar models or stepwise segment representations, which support the execution of the underlying operations without necessitating a deeper structural analysis of the relationships involved. These problems require that pupils recognize additive and multiplicative relationships between given data and an unknown, but they can often be solved without a deep structural understanding of algebraic relations. Following Stacey and MacGregor (1999) it can be argued that pupils must not only grasp the relationships between the unknown quantities embedded in the problem situation but also develop the ability to represent these relationships symbolically and to manipulate them within an algebraic framework. The ability to carry out such solutions proves that pupils possess an operational way of thinking on the constituent elements of the problem situation, but it does not necessarily reflect an understanding of

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the overall structure of the problem situation (Stacey & MacGregor, 1999). By contrast, problems modelled by $Ax + B = Cx + D$ equation present qualitatively different demands. Here, two expressions must be coordinated, often involving quantities that vary simultaneously. These tasks are not suitable to straightforward arithmetic modelling, as methods such as bar models prove inadequate for reflecting the relational symmetry between both sides of the equation. Their resolution requires a higher level of structural thinking, in which pupils must recognize and interpret the links between unknown values as they appear in the given problem situation, and also to develop the ability to represent these relationships symbolically and to manipulate them within an algebraic framework, and also integrate them into a coherent equation that captures the entire problem situation (Sfard, 1991; Kieran, 1992). Without this structural way of thinking, pupils often rely on "guess and check" strategies which, while sometimes effective in simpler cases, do not promote the algebraic understanding needed to progress. Since algebraic knowledge is essential for solving more complex problems in later studies, relying on guessing and checking alone is insufficient for sustained mathematical development. This distinction between “arithmetical” and “algebraic” problem types (modelled by the equations $Ax + B = C$ and $Ax + B = Cx + D$, respectively) aligns with Freudenthal’s theory (1973) of Realistic Mathematics Education (RME) provides a distinctive perspective for understanding the transition from arithmetic to algebra. Central to his approach is the idea of *guided reinvention*, whereby pupils are not simply presented with formal algebraic symbolism but are instead encouraged to progressively reconstruct mathematical concepts by organizing and mathematizing their own experiences. In this view, algebra emerges naturally from problem contexts that are meaningful and experientially real to pupils’ development. Freudenthal (1973) emphasized that the shift from arithmetic to algebra is not merely a technical extension but rather a fundamental change in the mode of thought: from operating on numbers to reasoning about relations. This change necessitates what he termed *progressive formalization* which is a gradual abstraction process in which pupils initially rely on concrete strategies (such as counting or visual models), then move toward more structured notations and, ultimately, symbolic algebra. The pedagogical task, therefore, is to design learning environments where pupils can mathematize contextual problems, gradually recognizing invariant structures that can later be representing by algebraic expressions and equations. Within this framework, arithmetic strategies retain an important role at the early stages, since they provide the experiential foundation upon which structural thinking can be built. However, Freudenthal (1973) cautioned that without a deliberate shift toward structural understanding, pupils may remain confined to procedural calculations, thereby missing the essence of algebra as a language of relationships. His model thus positions algebra not as a sudden curricular introduction but as the culmination of a carefully scaffolded trajectory of mathematization. As we have previously argued (Fülöp, 2025), the transition from arithmetic to algebra in lower secondary school education can be supported through a three-step model for a systematic development of structural thinking. The first stage involves the use of numerical methods, such as the method of false position (regula falsi), which allow pupils to experiment with specific numerical substitutions. This process not only facilitates solution attempts but also develops structural thinking, as pupils begin to perceive the underlying structure of the problem situation while testing concrete numerical values. The second stage introduces the functional approach to algebra: pupils are guided to interpret the ways in which unknown quantities relate to each other, viewed as mappings from one variable to another, thereby extending their conceptual resources beyond procedural calculations. Finally, in the third stage, pupils formalize these insights by formulating equations, recognizing that the solution consists in identifying the value of the unknown for which two algebraic expressions - corresponding to the left- and right-hand sides of the problem - attain equality. Taken together, these stages delineate a developmental trajectory whereby pupils progress from exploratory numerical reasoning, through functional interpretation, to algebraic formalization, thus laying a foundation for the structural way of thinking indispensable in algebra.

In the present study, this theoretical background provides the foundation for examining Grade 7 pupils’ approaches to word problems that vary in structural complexity. The primary aim of the present study is to investigate the extent to which Grade 7 pupils, prior to formal instruction in algebra, operate at different levels of procedural and structural thinking. Specifically, we seek to identify how far pupils rely on procedural approaches, sufficient for solving problems that can be modeled by equations of the form $Ax + B = C$, and to what degree they exhibit elements of structural thinking, which is indispensable for handling word-problems requiring algebraic models of the type $Ax + B = Cx + D$. This focus allows us to map pupils' reasoning before algebra is explicitly taught, thereby situating their approaches along a developmental continuum that ranges from procedural reasoning to fully structural competence. Procedural thinking refers to the capacity to process problems modeled algebraically as equations like $Ax + B = C$. Such tasks typically do not require a global understanding of the problem's structure, as they can often be solved through arithmetic strategies without the explicit formulation of an equation. This view aligns with with Kieran's (1992) distinction between arithmetic-based approaches and the emerging symbolic reasoning required in early algebra.

Structural thinking, by contrast, entails a holistic grasp of the problem situation, enabling the pupil to represent and solve problems that involve interdependent quantities, typically modeled by equations of the form $Ax + B = Cx + D$. The key feature of structural thinking is precisely this capacity to coordinate relationships across the entire task and to express them in algebraic symbolism, a shift that parallels Sfard's (1991) account of the reification process by which mathematical operations are transformed into objects of thought. The transition from procedural to structural thinking is so fundamental that Filloy and Rojano (1989) famously referred to it as a *cognitive gap*, emphasizing the conceptual leap required for pupils to progress. An additional perspective for interpreting the developmental trajectory from procedural to structural thinking can be drawn by comparing it with the SOLO taxonomy (Biggs & Collis, 1982), which describes levels of increasing complexity in pupils' responses. The two stages distinguished in the present framework - procedural and structural thinking - can be meaningfully aligned with the SOLO taxonomy levels. At the *procedural stage*, pupils are typically able to execute the necessary operations in solving problems which can be approached arithmetically without requiring an integrated understanding of the overall structure. This mode of reasoning is closely related to the *multistructural level* in the SOLO taxonomy: several elements of the problem are recognized and processed, but they are not yet coordinated into a coherent whole. At the same time, the *structural stage* is achieved when pupils can grasp the entire structure of the problem and model it algebraically. In the case of problems modelled by an $Ax + B = Cx + D$ equation, they succeed in coordinating all relevant relations within a single representation. This ability to organize and unify different aspects of the problem aligns well with the *relational level* of the SOLO taxonomy.

Beyond this, the SOLO taxonomy also identifies an *extended abstract level*, which involves generalizing and transferring reasoning structures to novel contexts. Although this level lies beyond the scope of Grade 7 pupils' Mathematics education, it aligns with the broader aims of early algebra instruction, where pupils are expected not only to solve particular word problems but also to develop general algebraic strategies.

Understanding this developmental trajectory - from simple procedural to fully structural thinking - provides an essential perspective for interpreting pupils' approaches to algebra-related word problems. As Radford (2010) has argued, the emergence of algebraic thinking cannot be reduced to symbol manipulation alone; rather, it involves the progressive coordination of representations, reasoning, and generalization. This perspective highlights the cognitive demands inherent in bridging arithmetic reasoning and formal algebraic modelling, and underscores why pupils' difficulties at this stage are not merely technical but deeply conceptual.

Determining the degree to which Grade 7 pupils from diverse schools operate at these levels is the primary aim of this investigation. By mapping pupils' responses onto these levels of reasoning, we seek to establish a more comprehensive overview of their readiness for algebra. Such knowledge is not merely diagnostic: it directly informs the design of instructional strategies. If pupils predominantly operate at a procedural level, then instruction must focus on scaffolding their transition toward relational understanding. Conversely, if pupils are beginning to demonstrate structural awareness, teaching can emphasize more explicit algebraic formalization. In this way, the study contributes both to the theoretical discourse on the arithmetic-to-algebra transition and to the practical task of designing effective algebra instruction at the lower secondary level. Ultimately, it is only by understanding the current developmental stage of pupils' structural reasoning that educators can design the progression of tasks and representations that reinforce algebraic thinking.

2. Research Methodology

2.1. Participants

The research was conducted at the Reformed High School in Gödöllő with 20 seventh-grade pupils during the 2024/2025 academic year. At the start of Grade 7, these students were enrolled in a specialized mathematics class, many of them demonstrated high mathematical performance. All classroom activities and assessments were administered by the author. Having completed their first six grades in various other schools, the pupils have shown a wide range of proficiency in solving word problems at the beginning of the school year. While most were familiar with conventional arithmetic methods, a few had already begun applying algebraic approaches to tackle such problems. Since the participating pupils originated from different schools, they had been exposed to a variety of teaching approaches and instructional methods regarding word problem solving. Consequently, their strategies for approaching such problems reflected the diverse pedagogical backgrounds of their teachers.

2.2. Research objectives and questions

The primary objective of the present study was to assess to what extent pupils rely on procedural thinking and to what degree they demonstrate the structural thinking required for more abstract tasks.

Based on the objectives mentioned above, the following research questions guided our research work, seeking to identify answers to the following issues:

- To what extent do seventh-grade pupils rely on procedural strategies when solving word problems that admit an arithmetic solution, and how does this reliance indicate the limits of their structural awareness?
- In what ways can pupils' successful solutions to problems of the form $Ax + B = Cx + D$ be taken as evidence of emerging structural thinking, beyond procedural manipulation?
- How do variations in prior instructional experiences across different schools shape the strategies that pupils employ when approaching both arithmetic- and algebra-related word problems?
- What patterns can be identified in the distribution of pupils across the categories of procedural, pre-structural, and structural thinking, and how do these patterns illuminate the cognitive demands of different task types?

2.3. Method

For this purpose, pupils were asked to solve a test paper containing a set of nine exercises within a time frame of 60 minutes. The task set was carefully designed to include most problems that could be solved using purely arithmetical strategies, thereby relying mainly on procedural reasoning. At the same time, in order to evaluate the pupils' emerging capacity for structural reasoning - an ability indispensable for the meaningful introduction of algebra - the test also incorporated a few exercises that can be modelled by equations of the form $Ax + B = Cx + D$. These latter problems were deliberately selected to require a higher level of abstraction, as they involve an underlying relational structure between quantities on both sides of the equation. It should be emphasized that none of the pupils had received systematic instruction in algebra at the time of the experiment. A few of them reported that their teachers had occasionally referred to algebraic formulations as a curiosity, but such exposure was fragmentary and informal. Therefore, the expectation was that most pupils would attempt even these more complex tasks through arithmetical reasoning (e.g., by interpreting part–whole and difference relations arithmetically). The successful solution of such tasks requires a higher level of structural reasoning, extending beyond procedural steps to encompass the recognition and coordination of relationships within the problem and an integrated understanding of its overall structure.

3. From strategy to structure: A qualitative analysis of pupils' solutions

In the following, we present a detailed qualitative examination of the pupil responses obtained throughout the teaching experiment, focusing on how pupils at the beginning of Grade 7 approach algebra-related word problems. Drawing on the theoretical framework introduced earlier, particularly the transition from procedural to structural thinking (Filloy & Rojano, 1989; Kieran, 2004; Blanton et al., 2015), we examine the extent to which individual solutions reflect varying levels of cognitive development. The selected pupil work is discussed not merely in terms of correctness, but as evidence of underlying reasoning strategies, use of representations, and levels of generalization. Attention is paid to how pupils navigate problem structure, symbolization, and the coordination of quantities. Within this conceptual framework, each solution is contextualized within existing research on early algebra learning and conceptual understanding, allowing for a nuanced interpretation of learning trajectories and instructional implications.

As we mentioned above, the test paper contained nine word-problems of different types, designed to elicit both procedural and structural reasoning, as follows.

1) A simple multiplicative comparison.
2) An additive comparison with two related quantities.
3) A price–quantity coordination problem.
4) An implicit algebraic relation, which can be modelled by an $Ax + B = C$ equation.
5) A problem with two unknowns in direct relation.
6) A word-problem modelled by an $Ax + B = Cx + D$ equation, requiring balance reasoning.
7) A subtractive comparison ('x less than y').
8) A multi-variable task which requires simultaneous reasoning.
9) A backward problem which requires inverse operations.

3. 1. Problems involving the assessment of multiplicative and additive reasoning

PROBLEM 1: *A farm has a total of 3850 animals, sheep and goats. The number of sheep is four times the number of goats. How many sheep and how many goats are there?*

This problem presents a multiplicative relationship between the two unknown quantities. 19 pupils gave the right answer: 17 using arithmetic methods (typically through segment line representation, or proportional reasoning and part-whole calculations), one using a linear equation, and one by trial-and-error. The only incorrect response was due to a computational mistake, rather than a conceptual misunderstanding. The high success rate suggests that pupils are generally comfortable dealing with multiplicative structures when the relationship is straightforward and aligns with previously learned partitive or quotative division strategies (Verschaffel et al, 2000). Most pupils relied on breaking the total quantity into five equal parts (1 part goat, 4 parts sheep) and scaling accordingly, a strategy supported by research into early algebraic reasoning via visual or schematic models (Blanton et al., 2015).

PROBLEM 2: *In a workshop, 2857 screws were produced over two days. On the first day, 345 more screws were made than on the second day. How many screws were produced on each day?*

This task required the interpretation of an additive relationship between two unknowns. 15 pupils solved it correctly: 14 using arithmetic decomposition strategies (often relying on equalizing the quantities and redistributing the difference), and one pupil by trial-and-error method. Five pupils, despite using arithmetic approaches, gave the wrong answer primarily due to misinterpretation of the additive relationship or procedural errors during decomposition. The most frequent erroneous approach followed a similar pattern across several pupils. They divided the total by two, obtaining

2857 ÷ 2 = 1428.5, and then proceeded to add or subtract the 345 adjustments. One pupil rounded these results to 1774 and 1483, thereby producing integer answers. However, other pupils left their responses in decimal form (1773.5 and 1483.5), without recognizing that such non-integer values are not meaningful within the problem context. This error pattern illustrates an important cognitive feature of procedural reasoning at this stage: pupils were able to perform the arithmetic operations correctly but failed to coordinate them with the constraints of the problem situation. The inability to reconcile the obtained numerical results with the requirement for whole-number solutions suggests that the structural dimension of the task — understanding the interdependence of quantities and their realistic interpretation — was not sufficiently engaged. This indicates that while procedural manipulation was successfully applied, the step toward structural thinking, which demands a consistency check between calculation and context, remained unaccomplished.

Compared to the first task, this problem posed greater difficulty, even though it involved a smaller numerical range. The difference may be attributed to the cognitive load imposed by additive comparison structures, which are often less intuitive for pupils at this stage of development (Carpenter, Franke, & Levi, 2003). While multiplicative problems can often be visualized using repeated groups, additive comparison problems require more abstract reasoning and a solid understanding of inverse operations. This finding aligns with research indicating that additive structures are often more challenging for pupils transitioning from arithmetic to algebraic thinking, particularly when no visual support (e.g., bar models or strip diagrams) is explicitly provided (Ng & Lee, 2009).

Both tasks are suitable to visual or schematic problem-solving strategies such as bar modelling or "strip diagrams" (e.g., the Singapore model), which are effective in bridging arithmetic and algebraic thinking (Cai & Knuth, 2011). The relatively high success rate in both tasks suggests that most pupils demonstrated strong procedural (operational) thinking, particularly in contexts that allow for direct manipulation of known quantities. However, the differential performance between the two problems reveals a key insight: while pupils can apply conventional arithmetic strategies to multiplicative problems with high confidence, additive comparison problems still present a significant challenge. This discrepancy underscores the need to systematically develop pupils' structural understanding specifically, the ability to recognize and represent relationships between unknown quantities, a foundational component of early algebraic thinking (Kieran, 2004).

PROBLEM 8: *Bea spent 5450 forints in three days. On the first day, she spent three times as much as on the second day, and on the third day, she spent 40 forints more than on the second day. How much did she spend on the first day?*

6 pupils gave the right answer (5 pupils by conventional arithmetic calculations, 1 pupil by trial-and-error method), 7 pupils gave wrong answer (5 pupils by arithmetic, 2 pupils by trial-and-error) and 7 pupils gave no response. The success rate on this problem solving was notably lower than on previous similar tasks (e.g., problems 1 and 2), despite its apparent structural similarity. This suggests that surface similarity is not sufficient to predict success, the underlying complexity of variable relationships plays a critical role. Despite involving only one unknown quantity (the second day's spending), the task requires pupils to coordinate three interrelated amounts: base quantity (second day), multiplicative relationship (first day) and additive relationship (third day). Even though the algebraic model is a linear equation of the form $Ax + B = C$ (so the task could be considered a typical arithmetical problem, which requires procedural thinking) the simultaneous use of both multiplicative and additive relationships seems to challenge many pupils. As highlighted by Nathan and Koedinger (2000), pupils' performance does not always align with teachers' expectations that arithmetic methods should be more accessible than algebraic ones. Their research demonstrated that pupils often find pure arithmetic word problems more challenging when these implicitly demand algebraic reasoning, such as the coordination of multiple representations or transformations. Therefore, the algebraic methods to solve the same problem in some cases may elicit higher success rates, precisely because the symbolic notation provides a clearer representation of the underlying structure of the problem situation (writing the relationships between data). In this sense, what might superficially appear to be the simpler,

arithmetic form of a problem can in fact impose a greater cognitive load, whereas the algebraic solution allows pupils to engage more directly with the relationships among the quantities involved. Furthermore, the low success rate (6 out of 20) suggests that the cognitive load of simultaneously tracking and interpreting three quantities (each defined in terms of a different mathematical relationship) is significant. Despite surface similarities with earlier tasks, PROBLEM 8 revealed deeper cognitive and conceptual demands due to the coordination of three relationally defined quantities. The low solution rate and high number of omitted responses point to the limits of procedural fluency and the growing importance of structural reasoning. Tasks like this act as valuable indicators of which pupils are beginning to shift toward an algebraic frame of mind, even before formal instruction in algebra begins.

3. 2. The role of reverse thinking in solving word problems

This task exemplifies a problem type that is primarily arithmetic in nature and can be efficiently approached through backward reasoning. Rather than requiring a comprehensive structural model, the solution depends on the ability to invert the sequence of operations and trace the problem situation in reverse. Such problems thus provide insight into pupils' procedural fluency and their capacity to apply flexible reasoning strategies within an otherwise straightforward arithmetic context.

PROBLEM 9: *In a town, the number of inhabitants doubled, and then 456 inhabitants moved away. After these changes, the town had 1230 inhabitants. How many inhabitants were there originally in the town?*

The results showed a rather clear dominance of arithmetic approaches. 11 pupils solved the problem correctly (10 through arithmetic reasoning working backwords and only 1 pupil wrote an algebraic equation), 3 pupils gave wrong answers (2 pupils through arithmetic, and 1 pupil through algebraic method), and 6 pupils did not attempt the problem at all. The correct arithmetic solutions generally followed a backwards reasoning strategy, reversing the operations described in the problem: starting from 1230, adding back 456, and then dividing by two. This method is consistent with findings by Verschaffel, Greer, and De Corte (2000), who emphasize that pupils often prefer informal, context-based strategies such as "undoing" when such approaches are efficient and readily applicable. This demonstrates that pupils, when solving word problems modelled by an equation of the form Ax+B=C, they can remain within the domain of procedural reasoning and succeed without appealing to explicit algebraic symbolism. Among the incorrect arithmetic solutions one pupil committed a simple calculation error, and the other pupil, however, followed the path 1230−456=1224, and then divided by 2, yielding 612. This error reflects a *partial structural understanding*: the pupil recognized the need to reverse operations but misinterpreted the order of the transformations. Stacey and MacGregor (1999) describe such errors as indicators of "local coherence" rather than global structural awareness: the pupil can manage segments of the problem but fails to integrate them into a coherent overall model. Such tendencies highlight that even when procedural reasoning is accessible, the lack of structural integration may lead to systematic misapplications of otherwise valid operations. One pupil wrote the correct equation 2x-456=1230, then divided by two in the following way: "x-456=615" and gave the solution "x=1071". This highlights a frequent difficulty identified in research (Kieran, 1992), namely that while pupils may be able to represent problem structures symbolically, procedural fluency in algebraic manipulation is often not yet secure. In this sense, the task sits at an interesting boundary: while arithmetic strategies suffice, an algebraic approach offers a more transparent representation of the underlying relational structure.

Finally, the fact that 6 pupils did not attempt the problem is also noteworthy. Despite the task being accessible via arithmetic reasoning, a non-negligible portion of the class either did not recognize a feasible entry point or perhaps experienced cognitive overload when parsing the two-step transformation. This is particularly surprising given that the strategy of working backwards, as well as tasks structurally similar to this one, are explicitly introduced and practiced in Grade 6, meaning that pupils should already be familiar with this type of problem as a rehearsed exercise. As Mayer (1985) emphasized, working backwards is one of the fundamental heuristics for arithmetic word problem solving, typically mastered during upper primary years. Similarly, Verschaffel, Greer, and De Corte

(1999) argue that when pupils fail to apply rehearsed strategies to familiar problem types, this often signals a gap between procedural rehearsal and the flexible transfer of strategies to slightly varied contexts.This aligns with Stacey and MacGregor's (1999) observations that word problems involving more than one varying quantity often overload pupils' working memory, particularly when they have not yet automatized strategies for reversing multi-step processes.

3.3. Problems involving relational reasoning and emerging structural demands

The following problems fall within a transitional space between tasks that can be solved through straightforward procedural strategies and those that necessitate fully developed structural reasoning. While they remain, at least in principle, amenable to arithmetic approaches, their successful solution increasingly requires pupils to recognize relational dependencies among quantities and to maintain an overview of the entire problem situation. As such, these solutions to these tasks highlight whether pupils are beginning to move beyond procedural reliance towards the emergent forms of structural thinking that constitute a necessary foundation for algebraic modelling.

PROBLEM 5: *Paul has 12 more stamps than twice the number of stamps Peter has. Together they have 168 stamps. How many stamps do they have separately?*

This problem was correctly solved by 11 pupils (6 answers by arithmetic methods, 4 answers through trial-and-error and 1 answer by algebraic equation). Meanwhile, 9 pupils made errors using arithmetic methods. We have to mention, that PROBLEM 5 has a linear arithmetic structure that is still accessible through arithmetic reasoning, particularly with visual strategies like bar modelling (Ng & Lee, 2009) or strip diagrams. The additive and multiplicative relationships are clearly laid out, and the numerical difference ("12 more") makes the logic intuitive for pupils relying on procedural methods. The relatively high success rate (compared to a more complex task like PROBLEM 7) suggests that even though the problem involves a two-variable relationship, its verbal formulation supports procedural thinking. This aligns with findings from Carpenter et al. (2003) and Blanton et al. (2015), who emphasize that certain problems, while technically algebraic in structure, remain solvable by primary school pupils through arithmetic methods because the relationship is familiar and concrete. The trial-and-error strategy used by several pupils also highlights a reliance on contextual reasoning and estimation, a practice that is often valid at this stage of early algebraic thinking (Cai & Knuth, 2011).

PROBLEM 7: *There are 1236 pupils in a school. The number of boys is 63 less than twice the number of girls. How many boys and girls are there, separately?*

Only 2 pupils gave the right answer, both using arithmetic reasoning. 14 pupils gave incorrect answers (12 with arithmetic methods, 1 with trial-and-error, 1 writing an equation), and 4 pupils gave no answer. In the comparison of pupils' approaches to PROBLEM 5 and PROBLEM 7, a clear contrast emerges in how pupils handle the relational structures involved. Task 5, which was based on the expression "12 more than," proved to be much more accessible for seventh-grade pupils. The addition-based relation could be approached not only through algebraic reasoning but also by means of straightforward arithmetic strategies, such as drawing bar models, making stepwise representations, or applying guess-and-check procedures. In the case of PROBLEM 7, the relationship expressed as "63 less than" poses a qualitatively greater challenge for pupils compared to the "12 more than" structure in PROBLEM 5. While the latter can often be effectively modeled using arithmetic strategies (such as the bar model or stepwise segment representation) the former cannot be adequately handled by these same strategies, as its structure requires a different, more advanced representational approach. The difficulty lies in the fact that subtractive comparative structures do not lend themselves as readily to additive reasoning patterns, and thus the common arithmetic heuristics become less transparent. A more efficient approach in PROBLEM 7 would be the algebraic method (i.e. writing an equation), which directly captures the dependency between the quantities. In such cases, setting up and solving an algebraic equation would provide a more straightforward and efficient approach, even though pupils at this stage have not yet fully developed these skills. Consequently, solving PROBLEM 7 requires a higher level of structural thinking, namely the ability to recognize and manipulate the underlying relational framework abstractly, whereas PROBLEM 5 could still be approached primarily at a procedural level. This shift marks a crucial cognitive transition point, consistent with Sfard's (1991) distinction between operational and structural conceptions, highlighting the cognitive demands that certain problem structures impose even before formal algebraic instruction is introduced. Furthermore, previous research works has demonstrated that problem structures involving negative relationships or "less-than" phrasing tend to be cognitively more demanding for pupils. One important reason is that such relations are less easily supported by intuitive or visual representations, which often serve as scaffolds in arithmetic-based reasoning (Verschaffel, Greer, and De Corte, 2000). According to Stacey and MacGregor (1997), pupils frequently misunderstand the symmetry of relational expressions such as "x less than y" often interpreting them in reverse or inconsistent ways, especially when lacking formal contextual cues. Their findings illustrate how natural language interference and limited symbolic understanding can lead to systematic errors. According to our findings, more than half of wrong responses may also be linked to the so-called *reversal error*, well-documented phenomenon in early algebra problem solving where pupils systematically invert relational statements. The reversal error has been identified at different levels of study, among high school pupils in their initial stage of formal algebra learning (MacGregor and Stacey, 1993). In addition, the phenomenon is complex because it involves the difficulties linked to the transition from arithmetic to algebra, as well as the general difficulties related to the change of representation from the natural to a formal language (González-Calero, Berciano and Arnau, 2020).

3.4. Towards structural coordination of interrelated quantities

PROBLEM 3: *Bea paid 2410 forints for a toy using 20 HUF and 50 HUF coins. She used five more 20 HUF coins than 50 HUF coins. How many coins of each type did she use?*

This problem introduces a more complex structure than the previous two tasks. It requires pupils to consider *two different attributes* for each unknown quantity: the number of coins and their monetary value. 10 pupils solved the problem correctly (7 using arithmetic strategies, and 3 through trial-and-error). Eight pupils gave wrong answers: 2 through trial-and-error, 5 using arithmetic strategies, and 1 pupil wrote the equation $20x+50x=2410$. Two pupils did not attempt the task at all. The increased difficulty of this type of problem is consistent with its underlying mathematical structure. Unlike the other problems, this problem involves a *dual constraint system*—a numerical relationship (the number of coins) and a value-based relationship (the total sum). Solving such problems through arithmetic requires pupils to coordinate two layers of reasoning simultaneously: the discrete quantity of each coin type, and the contribution of each to the total value. Solving this type of item requires more than carrying out a sequence of operations: pupils must recognize and coordinate the system of relationships between different quantities. This requires the presence of *structural thinking*, since pupils need to handle several interrelated conditions simultaneously. In this way PROBLEM 3 is a strong discriminator between pupils with surface-level procedural thinking strategies and those beginning to coordinate multiple constraints (which is a distinguishing feature of structural mathematical reasoning). It also aligns with Sfard's (1991) distinction between operational and structural conceptions: the data suggest that seventh graders largely remain at the operational (procedural) stage, without yet having achieved a full reification of the underlying structures. The persistence of procedural dominance thus reflects that the transition to structural thinking—crucial for algebraic reasoning—has not yet generally taken place at this level. According to Verschaffel et al. (1994), pupils often struggle when multiple relational structures must be integrated in the problem-solving process. Similarly, Stacey and MacGregor (1999) emphasize that many lower secondary school pupils encounter cognitive overload when required to represent relationships involving more than one varying quantity, which often leads to incomplete or incorrect solutions. The relatively high number of pupils who attempted solution by *trial-and-error* suggests that the practical context of the problem (handling money) encouraged some intuitive or experience-based approaches. This aligns with findings by Warren (2004), who suggested that realistic contexts can promote engagement but may not always lead to mathematically sound strategies unless explicitly supported. Without a representational framework, such as tables, bar models, or algebraic notation, pupils often resort to guess-and-check when faced with interdependent variables. These results underscore the pedagogical importance of gradually introducing dual-constraint problems in a supportive environment, possibly using representational tools (e.g., diagrams, tables) that bridge procedural and structural thinking.

3.5. From relational comparisons to global structural modelling

PROBLEM 4: *A bus travelled four times as far on the first day as on the second. How far did it travel on each of the two days if it travelled 135 km more on the first day than on the second?*

PROBLEM 6: *On one hike, we did a third of the whole way and 3 km more. Now we have 3 fifths of the full distance and 5 km to go. What is the total length of the trip?*

The analysis of pupils' performance on Tasks 4 and 6 reveals important insights into the transition from procedural to structural reasoning in early algebra learning. Both tasks share the characteristic that they can be modelled by equations containing the unknown on both sides, and are therefore, in a strict sense, algebraic in nature. However, their cognitive demands and the levels of abstraction they require from pupils differ substantially.

Task 4 can be represented by a relatively simple linear equation of the form $Ax+B=Cx$, where pupils must coordinate a multiplicative comparison ("four times as much") with an additive relationship ("135 more"). This problem type often allows for arithmetic strategies such as reasoning backwards, comparing quantities, or constructing sequential representations (English, 2004; Stacey & MacGregor, 1999). Indeed, of the ten correct solutions observed, nine were achieved through arithmetic reasoning and one through a trial-and-error approach. Three of them used segment representation, essentially visualizing the relationship as three equal intervals corresponding to the difference of 135 km. The rest of the successful pupils used the operation 135:3=45, thereby implicitly recognizing that the difference between the first- and second-day's distances represents three "shares" of the second day's journey. Although these solutions did not explicitly involve algebraic symbolism, they reveal an important latent structural awareness, especially the recognition that the difference corresponds to a multiplicative relationship. Incorrect responses (nine in total) also illuminate common pitfalls. Several pupils treated the difference additively rather than multiplicatively, showing what Stacey & MacGregor (1999) refer to as *local coherence*—they could handle isolated steps but failed to integrate them into a consistent structural model. The overall performance pattern suggests that Task 4 is highly amenable to arithmetic approaches, yet successful solutions depend on correctly interpreting the relational structure between the two quantities.

Task 6, by contrast, presents a substantially greater cognitive demand. Its algebraic model corresponds to the form $Ax+B=Cx+D$, where the unknown quantity is embedded in fractional relations ("one third of the total distance" and "three fifths of the total distance"), and appears on both sides of the equation. Although this type of problem can, in principle, be approached through arithmetic strategies, its successful resolution presupposes a higher level of structural reasoning. Specifically, pupils must achieve conceptual integration of part–whole relations with additive adjustments, a demand that goes beyond sequential computation and requires managing relational complexity through proportional reasoning (Kieran, 1992). Pupils are thus forced towards a more abstract mode of thinking in which the overall structure of the problem, rather than local numerical manipulations, must be apprehended and coherently integrated within a unified structural framework. The performance data reflect these observations. 11 pupils gave the right answer, all relying on arithmetic reasoning. Their solutions involved interpreting the whole–part relationships correctly: by coordinating the one-third and three-fifths fractions with the additional kilometres, they derived that 1/15 of the route equals 8 km, leading to the correct total. In this sense, arithmetic remained viable, but only through a more advanced level of structural reasoning: pupils had to integrate fractional decomposition with additive constants and ultimately reconcile two different expressions for the same unknown quantity. 8 pupils gave wrong answers, a proportion that underscores the considerable challenges embedded in this problem type. The primary difficulty lay in mismanaging the interplay of fractional parts and constants. One pupil chose not to attempt the task. This distribution of outcomes reflects a broader pattern in the literature: when both sides of an equation contain the unknown, arithmetical strategies become increasingly difficult (Vergnaud, 1983; Sfard, 1991). Correct arithmetic solutions to Task 6 required careful reasoning that in effect anticipated the algebraic principle of equivalence and the presence of structural thinking without symbolic notation.

The comparison between the two tasks highlights a critical developmental point in algebra learning. Whereas Task 4 can still be handled through procedural reasoning and local coherence (Stacey & MacGregor, 1999), Task 6 demands a deeper structural awareness. Pupils must recognize that the unknown occurs simultaneously in two fractional expressions, and that a solution can only be obtained by reconciling both sides of the relation - a way of thinking that underpins algebraic symbolism. Taken together, the analysis of Tasks 4 and 6 shows that while both problems can be approached arithmetically, they differ substantially in the level of structural reasoning they require. Task 4 primarily tests pupils' ability to interpret multiplicative difference relations, whereas Task 6 already requires them to reconcile two algebraic expressions for an unknown - a cognitive leap that signals readiness for formal algebra. The pupils' performances confirm that such problems are valuable diagnostic tools for assessing where learners stand in their development from procedural arithmetic strategies toward algebraic, structural thinking. The error patterns further highlight this distinction. In Task 4, mistakes largely reflected misinterpretations of multiplicative relationships or procedural reversal. In Task 6, however, errors stemmed from deeper conceptual difficulties: coordinating fractions, aligning additive constants, and integrating both into a coherent representation of the whole. These findings resonate with Freudenthal's (1973) argument that algebra requires a new mode of reasoning distinct from arithmetic, where relational structures must be made explicit rather than inferred from procedural manipulations.

4. Assessing the prevalence of procedural vs. structural thinking in problem solving

The analysis of pupils' responses across the nine tasks reveals meaningful patterns in relation to their position within the progression from *procedural thinking* to *structural thinking*. The classification was carried out with careful attention to the nature of the solutions recorded in pupils' written works, following the principles articulated in the theoretical framework. Importantly, a single pupil may appear in different categories depending on the task, as the categorization reflects local reasoning strategies rather than global ability.

The categorization of pupils' responses in the present quantitative analysis was conducted through a rigorous and systematic review of individual pupil work. Two experienced teachers independently examined each solution, and categorization decisions were based on clearly defined, objective criteria grounded in both prior research and pedagogical insight. This methodological approach was aimed at ensuring the reliability and validity of the classification process.

Central to our analytical framework was the differentiation between *procedural* and *structural* thinking. Responses were assigned to the *procedural thinking* category when pupils demonstrated a correct understanding of the relationships among the known and unknown quantities but failed to produce a correct final answer. These responses typically indicated partial comprehension and the application of surface-level strategies without a comprehensive grasp of the problem structure. Importantly, the nature of specific problems influenced categorization. Tasks 1, 2, and 9 were characterized by predominantly arithmetic structures that could be successfully approached without requiring deep insight into the underlying relational structure of the problem situation. Accordingly, even correct solutions to these items were classified under *procedural thinking*, as they could be completed using basic arithmetic reasoning rather than algebraic structuring. For the remaining tasks - those that required modelling interdependent quantities or involved more complex relational reasoning - correct answers were indicative of *structural thinking*. In these cases, pupils were expected not only to identify relevant relationships but also to synthesize them into a coherent and accurate solution strategy. Incorrect responses that nevertheless demonstrated a correct attempt to express the relationships between quantities — for example, using variables, verbal explanations, or visual representations — were classified as procedural. Although these pupils had partially grasped the structure of the problem, they had not yet reached the level of full structural understanding required for a correct solution. In a small number of instances, classification proved less straightforward due to ambiguous or borderline pupil solutions. In such cases, the two evaluators collaboratively discussed the response and reached a joint decision. While we acknowledge a limited degree of subjectivity in these exceptional cases, the small proportion of such responses ensures that any resulting influence on overall trends is minimal and does not significantly impact the reliability of the analysis.

Overall, the categorization process aimed to go beyond surface-level correctness and to probe the nature of the pupils' underlying mathematical reasoning. This fine-grained classification was instrumental in identifying patterns in the development of algebraic thinking and in mapping the progression from procedural to structural understanding.

The final categorization of pupil responses yielded the following distribution: 58.9% of responses were classified as indicative of *procedural thinking* (106 responses), 27.2% as reflective of *structural thinking* (49 responses), while 13.9% of the responses (25 in total) could not be classified due to either being blank or lacking sufficient evidence for categorization.

These results offer significant insight into pupils' cognitive positioning at the early stages of lower secondary algebra instruction. The predominance of procedural reasoning aligns with existing research works (Filloy & Rojano, 1989; Kieran, 2004), which highlights that pupils often enter algebra instruction relying heavily on arithmetic reasoning and step-by-step procedural strategies. The relatively lower proportion of responses demonstrating *structural thinking* reflects the more advanced cognitive demands involved in modelling and solving tasks with interdependent quantities. As discussed earlier, structural reasoning requires pupils to conceptualize the problem holistically, recognizing and operating on the underlying relationships between unknowns. The fact that over a quarter of responses were classified into this category is encouraging, as it shows that some pupils have begun to internalize the kind of *relational understanding* that supports algebraic generalization — a key developmental goal in early algebra education. The *unclassified responses*, though smaller in number, serve as a reminder that not all pupil thinking fits neatly into predetermined frameworks. In some cases, pupils left the item blank, which may suggest affective or motivational barriers, while in others, their reasoning was difficult to interpret even after careful analysis.

5. Conclusion and methodological implications

This study set out to explore the nature of pupils' thinking as they transition from arithmetic-based to algebraic approaches in solving word problems. The results of our qualitative and quantitative analyses indicate that a substantial majority of participating pupils possess a form of procedural thinking that enables them to apply basic arithmetic methods to contextual problems. This procedural competence, often grounded in step-by-step calculations and surface-level manipulation of numerical data, represents an essential, yet limited, foundation for future algebraic learning.

However, the findings also underscore a crucial developmental insight: without more advanced structural thinking, which entails recognizing and manipulating the underlying relationships between quantities, the algebraic interpretation of word problems remains inaccessible to most pupils. In line with this, only a smaller proportion of responses demonstrated evidence of structural insight, where pupils conceptualized problems holistically and formulated generalized or symbolic representations of quantitative relations. Consequently, the study highlights a key pedagogical imperative: prior to the formal introduction of algebra in Grade 7, instructional efforts must deliberately target the development of structural reasoning. As Freudenthal (1973) argued, mathematical structures should not be introduced prematurely; instead, pupils must be guided to "reinvent" these structures through rich, meaningful experiences that build upon their existing knowledge. Similarly, in an earlier research work, we have shown that algebraic thinking cannot be developed purely through symbolic manipulation, it requires the capacity to abstract and generalize from contextualized numeric experiences. Our findings further underscore this point by illustrating that the teaching of algebra is most effectively structured as a three-stage progression: beginning with numerical computations that ground students in concrete quantitative relationships, followed by a functional approach that fosters the interpretation of these relationships as mappings between variables, and culminating in the formal formulation of equations (Fülöp, 2025). Yerushalmy (2000), among others, has shown that visual models, verbal reasoning, and dynamic representations play a crucial role in helping pupils move beyond procedural habits and into structurally grounded mathematical thinking.

To support this developmental trajectory, educators are encouraged to design and sequence tasks that:

- Emphasize relational understanding over numerical computation,
- Incorporate open-ended problems that invite multiple representations,
- Provide opportunities for metacognitive reflection on solution strategies, and
- Utilize tools such as diagrams, tables, and informal algebraic notation to scaffold the shift toward generalization.

In summary, the findings point to the necessity of a carefully scaffolded instructional bridge between arithmetic reasoning and formal algebra. By foregrounding structural thinking in early algebra education, teachers can better prepare pupils for meaningful engagement with symbolic mathematics. It is our hope that the insights offered by this study will prove valuable to mathematics educators working in lower secondary education, supporting the development of pedagogical practices that are both theoretically grounded and responsive to pupils' actual thinking.

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